International Journal of Mathematics and
Computer Science, **17**(2022), no. 4, 1677–1683

On the exponential Diophantine equation $(p+2)^x + (2p+1)^y = z^2$, where $p, p+2$ and $2p+1$ are primes

Wachirarak Orosram[1], Sawian Jaidee[2], Wipawee Tangjai[3]

[1]Department of Mathematics
Faculty of Science
Buriram Rajabhat University
Buriram 31000, Thailand

[2]Department of Mathematics
Faculty of Science
Khon Kaen University
Khon Kaen 40002, Thailand

[3]Department of Mathematics
Faculty of Science
Mahasarakham University
Maha Sarakham 44150, Thailand

email: wachirarak.tc@bru.ac.th, jsawia@kku.ac.th, wipawee.t@msu.ac.th

(Received May 31, 2022, Accepted July 13, 2022)

Abstract

For given primes $p, p+2$ and $2p+1$ with $p \equiv 1 \pmod 4$, we study the Diophantine equation $(p+2)^x + (2p+1)^y = z^2$.

1 Introduction

Exponential Diophantine equations have attracted the interests of several mathematicians in the last decade, especially equations in the form $a^x + b^y =$

Key words and phrases: Exponential Diophantine equation, Catalan's conjecture, Bunyakovsky conjecture.
Corresponding Author: Wipawee Tangjai.
AMS (MOS) Subject Classifications: 11D61.
ISSN 1814-0432, 2022, http://ijmcs.future-in-tech.net

z^2, where x, y and z are non-negative integers and a, b are positive integers. The majority of the methods hinged on Catalan's conjecture [1] which was settled by Mihailescu [2] in 2004 and on elementary properties in number theory such as divisibility, congruence and unique factorization. Some references in the case that a and b are fixed integers can be seen in [3, 4, 5, 6, 7]. The case that a or b are in certain classes of integers are also studied. In 2012, Peker and Cenberci [8] studied the equation $(4^n)^x + p^y = z^2$, where p is an odd prime, $n \in \mathbb{Z}^+$ and x, y and z are non-negative integers. In 2014, Suvarnamani [9] proved that $(p, x, y, z) = (3, 1, 0, 2)$ is the unique non-negative integer solution of $p^x + (p+1)^y = z^2$, when p is an odd prime number. In 2015, Bacani and Rabago [10] showed that there are infinitely many non-negative integer solution for $p^x + q^y = z^2$, when p and q are twin primes. Later, Hoque [11] proved that there are exactly two solutions for $(M_{pq})^x + (M_{pq}+1)^y = z^2$, where $p, q \in \mathbb{Z}$ such that $p > 0$, $q > 1$ and $M_{pq} = p^q - 1$. In 2018, certain forms of $p^x + (p+2k)^y = z^2$, where $k = 2, 3, 4$ and p is a prime number, have been separately investigated under some conditions by Burshtein and Fernando [12, 13, 14, 15]. Note that the bases p and $p + 2k$ are called twin primes and Sexy primes when $k = 1$ and $k = 2$, respectively. More recently related works appeared in [16, 17, 18].

Bunyakovsky conjectured [19] that a non-constant polynomial $f(x)$ over rational integers produces infinitely many rational primes if the polynomial $f(x)$ satisfies the following statements: (i) its leading coefficient is a positive integer, (ii) the greatest common divisor of all coefficients of such a polynomial is 1, (iii) it must be an irreducible polynomial over $\mathbb{Z}$ and (iv) for each prime p, there exists $n \in \mathbb{Z}_p$ for which $f(n)$ is not divisible by p and $\mathbb{Z}_p$ is the set of integers modulo p. The conjecture hasn't been proven in case f is not linear.

Inspired by the approaches used in [12, 13, 14, 15], we continue to study the same type of the exponential Diophantine equation in different bases.

2 Main results

The following well-known result, called Catalan's conjecture, will play crucial role in obtaining our main results.

Proposition 2.1. *(Catalans conjecture) Diophantine equation* $a^x b^y = 1$, *has a unique solution* $(a, b, x, y) = (3, 2, 2, 3)$, *where* a, b, x *and* y *are integers such that* $\min\{a, b, x, y\} > 1$.

For a fixed odd integer b, we provide a necessary and sufficient conditions for the existence of a non-negative integer solution of $1 + (2b+1)^y = z^2$:

Lemma 2.2. *Let b be an odd integer. A Diophantine equation $1+(2b+1)^y = z^2$ has a non-negative integer solution if and only if $b = 2t^2-1$ for some $t \in \mathbb{Z}$.*

Proof. Let (y, z) be a non-negative integer solution of $1 + (2b+1)^y = z^2$. If $y = 0$, then $z^2 = 2$, which is impossible. If $y > 1$, then $z > 1$. By Catalan's conjecture there is no non-negative integer solution. If $y = 1$, then $z^2 = 2b+2$. So $z = 2\sqrt{\frac{b+1}{2}}$. This means that $\sqrt{\frac{b+1}{2}}$ is a non-negative integer as z is a non-negative integer. Consequently, $b = 2t^2 - 1$ for some $t \in \mathbb{Z}$.
Conversely, let $b = 2t^2 - 1$ for some $t \in \mathbb{Z}$. It is not hard to check that $(y, z) = \left(1, 2\sqrt{\frac{b+1}{2}}\right)$ are solutions to the equation $1 + (2b+1)^y = z^2$. □

Corollary 2.3. *Let p be an odd prime number. The Diophantine equation $1+(2p+1)^y = z^2$ has an integer solution if and only if $p = 2t^2-1$, for some $t \in \mathbb{Z}$.*

Lemma 2.4. *Let b be an odd integer. The Diophantine equation $(b+2)^x + 1 = z^2$ has a non-negative integer solution if and only if $b = t^2 - 3$, for some $t \in \mathbb{Z}$ where $t \geq 2$.*

Proof. Let (x, z) be a non-negative integer solution of $(b+2)^x + 1 = z^2$. If $x = 0$, then $z^2 = 2$, which is impossible. If $x > 1$, then $z > 1$. By Catalan's conjecture there is no non-negative integer solution. If $x = 1$, then $z^2 = b+3$. So $z = \sqrt{b+3}$. Consequently, $\sqrt{b+3}$ is an integer since z is an integer. This implies that $b = t^2 - 3$ for some $t \in \mathbb{Z}$.
Conversely, let $b = t^2-3$, for some integer $t \geq 2$. Then $(x, z) = \left(1, \pm\sqrt{b+3}\right)$ are two solutions to the equation $(b+2)^x + 1 = z^2$. □

Corollary 2.5. *Let p be an odd prime. The Diophantine equation $(p+2)^x + 1 = z^2$ has an integer solution if and only if $p = t^2 - 3$ for some $t \in \mathbb{Z}$ where $t \geq 2$.*

In Corollaries 2.3 and 2.5, we see that the non-linear polynomials $2x^2 - 1$ and x^2-3 satisfy the conditions $(i)-(iv)$ in Bunyakovsky conjecture. Therefore, if there exist infinitely many solutions for the Diophantine equation in Corollaries 2.3 and 2.5, then these solutions support the Bunyakovsky conjecture.

In Theorem 2.6, we use the same method of proof as in [15] to derive the result.

Theorem 2.6. *Let $p, p+2$ and $2p+1$ be prime numbers such that $\frac{p+1}{2}$, $p+3$ are square integers and $p \equiv 1 \pmod 4$. The only non-negative integer solutions of the Diophantine equation $(p+2)^x + (2p+1)^y = z^2$ are $(x, y, z) = \left(0, 1, 2\sqrt{\frac{p+1}{2}}\right)$ and $\left(1, 0, \sqrt{p+3}\right)$.*

Proof. Let (x, y, z) be non-negative integer solutions of $(p+2)^x + (2p+1)^y = z^2$. If $x = 0$ or $y = 0$, then $(x, y, z) = \left(0, 1, 2\sqrt{\frac{p+1}{2}}\right)$ or $\left(1, 0, \sqrt{p+3}\right)$. Suppose $x > 0$ and $y > 0$. We consider the following cases:

Case 1: x and y are even. Since z is even, it follows that $z^2 \equiv 0 \pmod 4$. Observe that $(p+2)^x \equiv 1 \pmod 4$ and $(2p+1)^y \equiv 1 \pmod 4$, which imply that $z^2 = (p+2)^x + (2p+1)^y \equiv 2 \pmod 4$ which is a contradiction.
Case 2: x and y are odd. Again, we note that $z^2 \equiv 0 \pmod 4$ as z is always even. Since $(p+2)^x \equiv -1 \pmod 4$ and $(2p+1)^y \equiv -1 \pmod 4$, it follows that $z^2 = (p+2)^x + (2p+1)^y \equiv 2 \pmod 4$ which is a contradiction.
Case 3: x is even and y is odd. Let $x = 2k, y = 2s+1$, for some $k \geq 1$ and $s \geq 0$. We have $(p+2)^{2k} + (2p+1)^{2s+1} = z^2$, which can be rewritten as

$$(2p+1)^{2s+1} = z^2 - (p+2)^{2k} = \left[z - (p+2)^k\right]\left[z + (p+2)^k\right].$$

Thus, there exist non-negative integers α, β that $(2p+1)^\alpha = z - (p+2)^k$ and $(2p+1)^\beta = z + (p+2)^k$, where $\alpha < \beta$ and $\alpha + \beta = 2s+1$. Then we immediately get

$$2(p+2)^k = (2p+1)^\alpha \left[(2p+1)^{\beta-\alpha} - 1\right].$$

In case $\alpha = 0$, we have $2(p+2)^k = (2p+1)^{2s+1} - 1$. If $s = 0$, then $(p+2)^k = p$ which is impossible. If $s \geq 1$, then we have

$$(p+2)^k = p\left[(2p+1)^{2s} + (2p+1)^{2s-1} + \cdots + (2p+1) + 1\right].$$

This implies that $p \,|(p+2)$ which is impossible as p and $p+2$ are different prime numbers. If $\alpha \geq 1$, then $(2p+1)\,|(p+2)$ which is also impossible.
Case 4: x is odd and y is even. Let $x = 2k+1$, $y = 2s$ for some $k \geq 0$ and $s \geq 1$. We have $(p+2)^{2k+1} + (2p+1)^{2s} = z^2$, which can be rewritten as

$$(p+2)^{2k+1} = z^2 - (2p+1)^{2s} = [z - (2p+1)^s]\,[z - (2p+1)^s]$$

Thus, there exist non-negative integers α, β such that $(p+2)^\alpha = z - (2p+1)^s$ and $(p+2)^\beta = z + (2p+1)^s$, where $\alpha < \beta$ and $\alpha + \beta = 2k+1$. Then

$$2(2p+1)^s = (p+2)^\alpha \left[(p+2)^{\beta-\alpha} - 1\right].$$

In case $\alpha = 0$, we have $2(2p+1)^s = (p+2)^{2k+1} - 1$. If $k = 0$, then $2(2p+1)^s - p = 1$ which is impossible. If $k \geq 1$, then

$$2(2p+1)^s = (p+2)^{2k+1} - 1 = (p+1)\left[(p+2)^{2k} + (p+2)^{2k-1} + \cdots + (p+2) + 1\right]$$

Since $p+1$ is even and $(p+2)^{2k} + (p+2)^{2k-1} + \cdots + (p+2) + 1$ is odd, it follows that $p+1 = 2(2p+1)^l$, for some integer l such that $0 \leq l < s$. If $l = 0$, then $p = 1$ which is impossible. If $1 \leq l < s$, then $2(2p+1)^l - p = 1$ which is impossible. If $\alpha \geq 1$, then $(p+2)\,|(2p+1)$ which is impossible since $p+2$ and $2p+1$ are different primes. □

Following is an equivalent statement to Theorem 2.8:

Theorem 2.7. *Let $p, p+2$ and $2p+1$ be prime numbers such that $p \equiv 1 \pmod 4$. A Diophantine equation $(p+2)^x + (2p+1)^y = z^2$ has a non-negative integer solution if and only if $\frac{p+1}{2}$ or $p+3$ are square integers.*

3 Conclusion

By direct computation, we find that there is no a prime $p < 3 \times 10^5$ satisfying

i) $p+2$ and $2p+1$ are prime numbers,

ii) $p \equiv 1 \pmod 4$,

iii) $\frac{p+1}{2}$ or $p+3$ are square integers.

By Corollary 2.7, we conclude that the Diophantine equation $(p+2)^x + (2p+1)^y = z^2$ has no non-negative integer solutions when $p, p+2$ and $2p+1$ are prime numbers such that $p < 3 \times 10^5$. Thus, we conjecture that this equation has no non-negative integer solutions for any primes p such that $p+2$ and $2p+1$ are primes.

Acknowledgment. This research project was financially supported by Mahasarakham University. The authors wish to thank the referees for their kind suggestions and comments to improving this article.

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