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On metric-location-domination number of graphs

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Abstract

Brigham et al. (2003) derived some lower and upper bounds of the metric-location-domination number $\gamma_M(G)$ of any connected graph G; namely, $\max\{\beta(G), \gamma(G)\} \leq \gamma_M(G) \leq \min\{\beta(G) + \gamma(G), n-1\}$. In this paper, we discuss graphs G which attain the lower or upper bounds.

1 Introduction

Let $G = (V, E)$ be a simple connected graph. A set $D \subseteq V$ is said to be a *dominating set* of G if for every $u \in V - D$ there exists a vertex $d \in D$ so that $d(u, d) = 1$. The *dominating number* $\gamma(G)$ of G is the size of a minimum dominating set of G [8, 10]. For an ordered subset $R = \{r_1, \ldots, r_p\}$ of V and $v \in V$, the *representation* of a vertex v with respect to R is the p-vector $(d(v, r_1), d(v, r_2), \ldots, d(v, r_p))$, where $d(x, y)$ is the distance between vertices x and y in G. A set R is called a *resolving set* of G if all vertices of G have distinct representations with respect to R. The *metric dimension* of a graph G, denoted by $\beta(G)$, is the size of a minimum resolving set of G [1, 3, 5, 9].

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In this paper, we study the combination of the dominating set and the resolving set in graphs. If a dominating set D of a graph G is also a resolving set of G, then D is called a *metric-locating-dominating set* of the graph G. The *metric-location-domination number* of G, denoted by $\gamma_M(G)$, is the size of a minimum metric-locating-dominating-set of G. The concept of the metric-location-domination number of a graph was introduced by Henning and Oellermann [6] and Brigham et al.[2], independently. Brigham et al. [2] named it the resolving dominating set of a graph.

The characterization studies of all graphs of order n with the metric-location-domination-number $n-1$ or $n-2$ have been done by Henning and Oellermann [6] who also determined the metric-location-domination-number of all trees T as follows:

$$\gamma_M(T) = \gamma(T) + \ell'(T) - |S'(T)|,$$

where $S'(T)$ and $\ell'(T)$ denote the set of strong support vertices in a tree T and the total endpoints (vertex of degree one) in T that are adjacent to a strong support vertex (i.e., a vertex adjacent to at least two endpoints) of T, respectively.

For any graph G of order n, Brigham et al. [2] showed that:

$$\max\{\beta(G), \gamma(G)\} \leq \gamma_M(G) \leq \min\{\beta(G) + \gamma(G), n-1\}. \tag{1.1}$$

In this paper, we study graphs with the metric location domination number attaining the above lower or upper bound. We derive the properties of such graphs and give some classes of such graphs attaining the bound.

We follow some terminologies and notations for trees used in the paper of Slater [9]. Let v be a vertex of a tree T. A *branch* of T at v is defined to be a maximal subtree containing v as an endpoint. That is, a branch of T at v is a subgraph induced by v and one of the components of $T-v$. If v has degree d, then v has d different branches. A *branch path* of T at v (if $d(v) \geq 3$) is a branch isomorphic to a path starting from v. The vertex v is called a *stem* of the branch path at v. If v is a stem of at least one branch path, then the subgraph of T consisting of v and all its branch paths will be called a *leaf with the stem* v. Thus, a leaf containing b branch paths is homeomorphic to $K_{1,b}$, or we can say it is isomorphic to a subdivision of $K_{1,b}$. Note that every two leaves are disjoint. The following theorem of the metric dimension of a tree was proved by Slater [9]. Independently, it was also shown by Harary and Melter [5].

Theorem 1.1. *[9] Let T be a tree with a set V_1 of endpoints with $|V_1| \geq 3$. Let $L_1, L_2, \cdots, L_k$ be all the leaves of T, and let a_i be the number of branch* *paths in T that are in L_i. Then $\beta(T) = |V_1| - k$, and S is a minimum resolving set of T if and only if it consists of exactly one vertex from each of exactly $a_i - 1$ of the branch paths of L_i for each $i \in [1, k]$.*

2 Main Results

In this paper, we study all graphs G with the metric location domination number attaining the lower or the upper bound of Eq. (1.1). The discussion is divided into two subsections. The first subsection deals with graphs attaining the upper bound and the second subsection deals with of graphs attaining the lower bound.

2.1 Graphs attaining the upper bound

In this subsection, we study graphs G which attain the upper bound of Eq. (1.1); namely, $\gamma_M(G) = \beta(G) + \gamma(G)$.

In general, if R and D are a resolving set and a dominating set of G, respectively, then $C = R \cup D$ is a metric-locating-dominating set of G. Consequently, $|C| = |R| + |D| - |R \cap D|$. Furthermore, for a graph G, $\gamma_M(G) = \beta(G) + \gamma(G)$ if and only if for every minimum resolving set R and for every minimum dominating set D we have $R \cap D = \emptyset$.

In particular, we characterize all trees with the metric location domination number attaining the upper bound, as stated in Theorem 2.1.

Theorem 2.1. *Let T be a tree other than a path. Let $L_1, L_2, \cdots, L_p$ be all its leaves with $p \geq 1$. Then $\gamma_M(T) = \beta(T) + \gamma(T)$ if and only if L_i is either isomorphic to a star or a path for each i.*

Proof. Let T be a tree other than a path with all its leaves $L_1, L_2, \cdots, L_p$. Let L_i be either isomorphic to a star or a path. Let x_i be a stem vertex of L_i and let a_i be the number of branch paths in T that are in L_i. Let A be any minimum resolving set of T. By Theorem 1.1, the set A must consist of exactly one vertex from each of exactly $a_i - 1$ of the branch paths of L_i. In this case, A consists of exactly $a_i - 1$ endpoints of each L_i if $L_i \cong K_{1,n_i}$ or 0 if $L_i \cong P_m$, for some $m \geq 2$. Now, let B be a minimum dominating set of T. For a contradiction, suppose $A \cap B \neq \emptyset$ and $u \in A \cap B$. Then u must be adjacent to a stem x_i with $L_i \cong K_{1,n_i}$, for some i. Since there exists an endpoint y in L_i which is not in A, to dominate y in T we need to have either $y \in B$ or $x_i \in B$. Therefore, $\{u, y\} \subseteq B$ or $\{u, x_i\} \subseteq B$. Now consider

$B' = B - \{u, y\} \cup \{x_i\}$. Then B' is a dominating set of T and $|B'| < |B|$, a contradiction. Therefore, $A \cap B = \emptyset$. This implies that $A \cup B$ is a minimum metric-locating-dominating set of T, and $\gamma_M(T) = \beta(T) + \gamma(T)$.

Now, we prove the reverse direction. Let T be a tree other than a path with its leaves $L_1, L_2, \cdots, L_p$ and $\gamma_M(T) = \beta(T)+\gamma(T)$. For a contradiction, suppose there was a leaf L_i in T other than a star and a path, for some i. Thus there is a branch path $J := u_1, u_2, \cdots, u_{p-1}, u_p$ of L_i with $p \geq 3$ and $u_1 = x_i$. Therefore, vertex u_{p-1} must be in **any** minimum dominating set A of T. Since u_{p-1} is in a branch path of L_i, we can have a minimum resolving set B of T containing u_{p-1}. This implies that $x \in A \cap B$. Now, consider $S = A \cup B$. Then S is a minimum metric-locating-dominating set of T and $|S| \leq |A| + |B| - 1 < |A| + |B|$. Thus $\gamma_M(T) < \beta(T) + \gamma(T)$, a contradiction. Therefore, each leaf L_i is either a star or a path. □

Note that the equality in Theorem 2.1 was proved by Gonzalez et al. [4] in terms of strong support vertices and the number of end points adjacent to them. These terms were introduced by Henning and Oellermann [6] when they showed the metric-location-domination number of any tree in general. In Theorem 2.2, we express it in a different way.

Now, let T be a tree with all the leaves $L_1, L_2, \cdots, L_p$. For each leaf L_i other than a path, define l_i as the number of branch paths of lengths at least 2 in L_i. If L_i is a path, we define $l_i = 0$. Let X be the set of stems with no branch paths of length one. Let $k = l_1 + l_2 + \ldots + l_p - |X|$. Then, we have the metric-location-domination number of any tree as follows:

Theorem 2.2. *For any tree T, $\gamma_M(T) = \beta(T) + \gamma(T) - k$.*

Proof. Let T be a tree with leaves $L_1, L_2, \cdots, L_p$. Let x_i be a stem vertex of L_i and let a_i be the number of branch paths in in L_i. Let A be a minimum resolving set of T. By Theorem 1.1, the set A must consist of exactly one vertex from each of exactly $a_i - 1$ of the branch paths of L_i. For $1 \leq i \leq p$ and $1 \leq j \leq l_i$, let $J_{ij} := u_{i,1}, u_{i,2}, \cdots, u_{i,p_j-1}, u_{i,p_j}$ be a branch path of length at least 2 of L_i other than a path with $p_j \geq 3$ and $u_{i,1} = x_i$. Then A can be chosen so that $u_{i,p_j-1} \in A$, for all i and j. Of course, in this case, a leaf L_i isomorphic to a path in T will not give any contribution to the membership of A.

Now, let B be a minimum dominating set. Then, each u_{i,p_j-1} must be in B. Therefore, $A \cup B$ is a metric-locating-dominating set of T and $|A \cap B| = k$. Moreover, $A \cup B$ is a minimum metric-locating-dominating set of T by the choice of A. Thus $\gamma_M(T) = \beta(T) + \gamma(T) - k$. □

The *corona product* of two graphs G and H, denoted by $G \odot H$, is defined as a graph formed by taking $|V(G)|$ copies of graph H and connecting the i^{th} vertex of G to all vertices of the i^{th} copy of H. Iswadi et al. [7] investigated the metric dimension of the corona product $G \odot H$, for any graphs G and H. The metric dimension of $G \odot H$ depends on the structure of H; namely, if H has a dominant vertex (which is adjacent to any other vertices), then $\beta(G \odot H) = |V(G)|\beta(H)$, and $\beta(G \odot H) = |V(G)|\beta(K_1 + H)$ otherwise.

Theorem 2.3. *For any graph H, $\gamma_M(K_1 \odot H) = \beta(K_1 \odot H) + a$, with $a = 0$ or 1. Moreover, if $a = 1$, then for any minimum resolving set R of $K_1 \odot H$ there is a unique vertex v in H which is not dominated by R.*

Proof. Let x be a vertex adjacent to all vertices of H in $K_1 \odot H$. Let A be a minimum resolving set of $K_1 \odot H$. If $x \in A$, then there is at least one vertex $x_1 \in H$ adjacent to x and all other vertices of H. Then, in this case, A is also a dominating set of $K_1 \odot H$, and so $\gamma_M(K_1 \odot H) = \beta(K_1 \odot H)$. Now, if $x \notin A$ and A is a dominating set of $K_1 \odot H$, then $\gamma_M(K_1 \odot H) = \beta(K_1 \odot H)$. But, if A is not dominating, then $A \cup \{x\}$ is a dominating set of $K_1 \odot H$. Thus, $\gamma_M(K_1 \odot H) = \beta(K_1 \odot H) + 1$. Furthermore, since A is not a dominating set of $K_1 \odot H$, there is a vertex $v \in H$ not dominated by A. This means that v is not adjacent to any vertex of A and so $r(v|A) = (2, 2, \ldots, 2)$ since the diameter of $K_1 \odot H$ is 2. Such a vertex v must be unique since A is a resolving set of $K_1 \odot H$. □

Since the value of $\gamma_M(K_1 \odot H)$ is equal to either $\beta(K_1 \odot H)$ or $\beta(K_1 \odot H) + 1$, we define a graph H as a *special* graph if $\gamma_M(K_1 \odot H) = \beta(K_1 \odot H) + 1$. It is easy to see that $\overline{K}_n$ with $n \geq 2$ is a special graph since $\gamma_M(K_1 \odot \overline{K}_n) = \beta(K_1 \odot \overline{K}_n) + 1$, but K_n is not a special graph with $n \geq 1$.

Theorem 2.4. *Let G be a connected graph. For any graph H, $\gamma_M(G \odot H) = \beta(G \odot H) + \gamma(G \odot H)$ if and only if one of the following statements holds:*

(i) *H is a special graph,*

(ii) *$\gamma_M(K_1 \odot H) = \beta(K_1 \odot H)$ and there is a minimum metric-locating-dominating set L of $K_1 \odot H$ containing x and $L - \{x\}$ is not a dominating set of $K_1 \odot H$, where x is a vertex adjacent to all vertices of H.*

Proof. Let G be any connected graph and let H be any graph. Let x be a vertex adjacent to all vertices of H in $K_1 \odot H$ and let $x_1, x_2, \cdots, x_n$ be the corresponding vertices x in the i^{th} copy of $K_1 \odot H$ of $G \odot H$, where n is the order of G.

($\Rightarrow$) Let G be a connected graph. For any graph H, let $\gamma_M(G \odot H) = \beta(G \odot H) + \gamma(G \odot H)$. By Theorem 2.3, $\gamma_M(K_1 \odot H) = \beta(K_1 \odot H) + a$, with $a = 0$ or 1. If $a = 1$, then statement (i) holds. Now, let $a = 0$; namely, $\gamma_M(K_1 \odot H) = \beta(K_1 \odot H)$. Assume that every minimum metric-locating-dominating set L of $K_1 \odot H$ does not contain x, where x is a vertex adjacent to all vertices of H in $K_1 \odot H$. Then $L_{all} = L_1 \cup L_2 \cup \cdots \cup L_n$ is a minimum metric-locating-dominating set of $G \odot H$, with L_i being the corresponding metric-locating-dominating set L of the i^{th} copy of $K_1 \odot H$ in $G \odot H$. The set L_{all} is also a minimum resolving set $G \odot H$. Thus, $\gamma_M(G \odot H) \neq \beta(G \odot H) + \gamma(G \odot H)$, a contradiction. Therefore, in this case, there is a minimum metric-locating-dominating set L of $K_1 \odot H$ containing x, with x being a vertex adjacent to all vertices of H. But, if $L - \{x\}$ is a dominating set of $K_1 \odot H$, then $L_{all} = (L_1 \cup L_2 \cup \cdots \cup L_n) \backslash \{x_1, x_2, \cdots, x_n\}$ is a minimum metric-locating-dominating set of $G \odot H$, a contradiction. Therefore, there is a minimum metric-locating-dominating set L of $K_1 \odot H$ containing x and $L - \{x\}$ is a not dominating set, with x being a vertex adjacent to all vertices of H.

($\Leftarrow$) Let H be a special graph; namely, $\gamma_M(K_1 \odot H) = \beta(K_1 \odot H) + 1$. If A is a minimum resolving set of $K_1 \odot H$, then A is not dominating. Thus $x \notin A$ and $A \cup \{x\}$ is a metric-locating-dominating set of $K_1 \odot H$. In $G \odot H$, let A_i be the corresponding set A in the i^{th} copy of $K_1 \odot H$, for each i. Let $A = A_1 \cup A_2 \cup \cdots \cup A_n$. We show that A is a resolving set of $G \odot H$. Let u and v be any distinct vertices of $G \odot H$. If both vertices are in the i^{th} copy of $K_1 \odot H$, then u and v can be resolved by some vertex in A_i. If u is in the i^{th} copy of $K_1 \odot H$ and if v is in the j^{th} copy of $K_1 \odot H$ with $v \neq x_j$ and $i \neq j$, then there is a vertex $a \in A_i$ such that $d(u, a) \leq 2$ and $d(w, a) \geq 3$. Thus $r(u|A) \neq r(w|A)$. If $v = x_j$ in j^{th} copy of $K_1 \odot H$, then clearly some vertex b in A_j will resolve u and v. Therefore, A is a resolving set of $G \odot H$.

Next, we show that A is a minimum resolving set of $G \odot H$. For a contradiction, suppose there is a resolving set R of $G \odot H$ with $|R| = |A_1 \cup A_2 \cdots \cup A_n| - 1$. Then there is an i^{th} copy of $K_1 \odot H$, for some i such that the number of vertices of R in this copy is less than $|A_i|$. Then there are two vertices in this copy which are not resolved by R, a contradiction. Thus A is a minimum resolving set of $G \odot H$. Since $A \cup \{x_1, x_2, \ldots, x_n\}$ is a minimum metric-locating-dominating set of $G \odot H$, $\gamma_M(G \odot H) = \beta(G \odot H) + \gamma(G \odot H)$.

Now, let $\gamma_M(K_1 \odot H) = \beta(K_1 \odot H)$ and there is a minimum metric-locating-dominating set A of $K_1 \odot H$ containing x and $A - \{x\}$ is not a dominating set of $K_1 \odot H$, where x is a vertex adjacent to all vertices of H. Then A is also a minimum resolving set of $K_1 \odot H$. Now, consider $R = A_1 \cup A_2 \cup \cdots \cup A_n$, $B = \{x_1, x_2, \ldots, x_n\}$ and $L = R - B$, where A_i is the corresponding set A in the i^{th} copy of $K_1 \odot H$ in $G \odot H$. Clearly, B is a minimum dominating set of $G \odot H$.

Next, we will show that L is a minimum resolving set of $G \odot H$ provided $x_i \in A_i$ for each i. Let u, v be two distinct vertices of $G \odot H$. If both vertices are in the i^{th} copy of $K_1 \odot H$ and none of them is x_i, then clearly they will be resolved by $A_i - \{x_i\}$. If one of them is x_i, then both will be resolved by a vertex $b \in A_j$ for $j \neq i$. Now, let u be in the i^{th} copy of $K_1 \odot H$ and v be in the j^{th} copy of $K_1 \odot H$ in $G \odot H$ and $i \neq j$. If $u = x_i$ and $v = x_j$, then there is a vertex $b \in A_j$ such that $d(v, b) = 1 < d(u, b)$. If $u = x_i$ and $v \neq x_j$, then there is a vertex $a \in A_i$ such that $d(u, a) = 1 < d(v, b)$. In all cases, u and v are resolved.

Now, we will show that R is a minimum metric-locating-dominating set of $G \odot H$. Suppose there is a minimum metric-locating-dominating set M of $G \odot H$ with $|M| < |R|$. Then there is an i^{th} copy of $K_1 \odot H$ in $G \odot H$ such that the number of vertices of M in this copy is less than $|A_i|$. Since $\gamma_M(K_1 \odot H) = \beta(K_1 \odot H)$, there is a vertex in this copy which is not dominated by M, a contradiction. Therefore, R is a minimum metric-locating-dominating set of $G \odot H$. □

The question of whether graph H is special becomes interesting. What class of graphs satisfies this specific property? The following theorem gives some classes of such graphs. A vertex u in a graph H is called an *earring* if u is adjacent to at least two vertices of degree one. The *earring degree* of u, denoted by $d_e(u)$, is the number of neighbors of u with degree one. A vertex v in a graph H is said to be *full degree* if $d(v) = |V(H)| - 1$.

Theorem 2.5. *If a graph H contains earrings but no full degree vertex, then H is a special graph.*

Proof. Let H be a graph with b (≥ 1) distinct earrings, but no full degree vertex. Consider the graph $G = K_1 \odot H$. Let x be a vertex adjacent to all vertices of H in G. Let A be a minimum resolving set of G. Then there is exactly one earring u such that $u \notin A$ and A contains exactly $d_e(u) - 1$ neighbors of u with degree one in H. On the other hand, we know that $\{x\}$ is a dominating set of G. Therefore, by the minimality of A, we always have that $A \cup \{x\}$ is a minimum metric-locating-dominating set of G. Thus $\gamma_M(K_1 \odot H) = \beta(K_1 \odot H) + 1$ and so H is a special graph. □

There are many graphs containing no earrings, but they are special graphs. For instance, it can be easily verified that, for $n \geq 5$, paths P_n are special if $n \equiv 0, 3 \mod 5$, and cycles C_n are special if $n = 5$ or $n \equiv 1, 3 \mod 5$.

Theorem 2.6. *If a graph H contains a full degree vertex, then H is not a special graph.*

Proof. Let $u_1, u_2, \cdots, u_p$ be the full vertices of H, with $p \geq 1$ and H has no earrings. Consider the graph $G = K_1 \odot H$. Let x be the only vertex which is adjacent to all vertices of H in G. Note that $C = \{x, u_1, u_2, \cdots, u_p\}$ forms a clique in G. Let A be any minimum resolving set of G. Then p vertices in C must be in A. As a consequence, A also becomes a minimum metric-locating-dominating set of G. Thus $\gamma_M(K_1 \odot H) = \beta(K_1 \odot H)$ and so H is not a special graph. □

2.2 Graphs attaining the lower bound

In this section, we are going to study graphs attaining the lower bound of Eq. (1.1); namely, the graphs G with $\gamma_M(G) = \max\{\beta(G), \gamma(G)\}$. This lower bound is attained by a graph G if

(i) there is a resolving set R which is a subset of a **minimum** dominating set D of the graph G; or

(ii) there exists a dominating set D of graph G which is a proper subset of a **minimum** resolving set R of graph G.

Furthermore, if the condition of (i) holds, then clearly $\beta(G) \leq \gamma(G)$, and if the condition of (ii) holds, then we have that $\gamma(G) < \beta(G)$. The graph G attaining the lower bound is said to be *type L1* if the first condition holds, otherwise it is called *type L2*.

Theorem 2.7. *All trees without earrings other than P_3 are of type L1.*

Proof. Let T be a tree without earrings. If T is a path P_n on n vertices ($n \neq 3$ and $n = 3k + a$), then $D = \{x_{2+3i} \mid i = 0, 1, \cdots, k-1\} \cup \{x_n\}$ and $D = \{x_{2+3i} \mid i = 0, 1, \cdots, k\}$ are the minimum dominating sets of P_n if $a = 1, 2$ and $a = 0$, respectively.

Let $L_1, L_2, \cdots, L_p$ be all leaves of T. Since T has no earrings, each L_i will be not isomorphic to a star $K_{1,t}$, for some $t \geq 2$. For $i = 1, 2, \cdots, p$, let x_i be a stem vertex of L_i and let n_i be the number of branch paths in L_i.

Let $\ell_{i,1}, \ell_{i,2}, \cdots, \ell_{i,n_i}$ be the ordered branch paths of L_i with $|\ell_{i,1}| \geq |\ell_{i,2}| \geq \cdots \geq |\ell_{i,n_i}|$ and let $z_{i,1}, z_{i,2}, \cdots, z_{i,n_i}$ be the vertices of degree one in branch paths in L_i. Since T has no earrings, for each L_i there is at most one j such that $|\ell_{i,j}| = 2$ and, in this case, $j = n_i$ (if any) by the above ordering.

Now, define $R = \{u \sim z_{i,t} \mid 1 \leq i \leq p, 1 \leq t \leq n_i - 1\}$. By Theorem 1.1, R is a minimum resolving set of T. Since every vertex of degree one in T must be dominated by some vertex in a minimum dominating set of T, the set R can be extended to be a minimum dominating set of T. Therefore, $R \subseteq D$, for some minimum dominating set D of T. Since D is both a minimum resolving set and a dominating set of T, T is type L1. □

Note that, by Theorem 2.7, a path P_n is of type L_1 with $n \geq 4$.

Let G, H be two graphs. The graph $G + H$ is defined as a graph with the vertex set $V(G) \cup V(H)$ and the edge set $E(G) \cup E(H) \cup \{xy \mid x \in V(G), y \in V(H)\}$. In the next theorem, we provide a class of graphs of type L2.

Theorem 2.8. *Let G be any graph (possibly vacuous). For any integer $n \geq 2$, the graph $K_n + G$ is of type L2.*

Proof. Let G be any (possibly vacuous) graph. It is easy to see that, for any integer $n \geq 2$, $\gamma(K_n + G) = 1$ since any vertex of K_n dominates all other vertices in $K_n + G$. On the other hand, any resolving set of $K_n + G$ must contain one vertex from K_n. Therefore, in this case, there is a dominating set of graph $K_n + G$ which is strictly a subset of a minimum resolving set of graph $K_n + G$. Thus $K_n + G$ is of type L2. □

Chartrand et al. [3] characterized all graphs of order n having metric dimension $1, n-2$ or $n-1$. A path P_n is the only graph with metric dimension one, and a complete graph K_n $(n \geq 2)$ is the only graph with metric dimension $n-1$. A graph G of order n has $\beta(G) = n-2$ if and only if $G = K_{s,t}$ $(s, t \geq 1)$, $G = K_s + \overline{K_t}$ $(s \geq 1, t \geq 2)$, or $G = K_s + (K_1 \cup K_t)$ $(s, t \geq 1)$. By Theorem 2.8, the graphs K_n, $G = K_s + \overline{K_t}$ and $G = K_s + (K_1 \cup K_t)$ are of type L2. It is easy to see that $G = K_{s,t}$ $(s, t \geq 1)$ is of type L2, too. Furthermore, we can have a more general class of graphs of type L2 as the following theorem shows.

Theorem 2.9. *Let H be a graph of order $n \geq 2$ and $1 \leq k < n$. Let $V_1 \cup V_2 \cup \cdots \cup V_k$ be any partition of $V(H)$. Let G be a graph constructed from H by joining K_{n_i} $(n_i \geq 2)$ with all vertices of V_i for each i. Then the graph G is of type L2.*

Proof. Let $D = \{x_1, x_2, \cdots, x_k\}$, where $x_i \in K_{n_i}$. Then D is a minimum dominating set of G since $k < n$ and we have to dominate all vertices of H and each K_{n_i}. On the other hand, if R is minimum resolving set of G, then R must contain $n_i - 1$ vertices for K_{n_i} in each i. So we can have a minimum resolving set R of G which contains all vertices of D; namely, $D \subset R$. Therefore, G is of type L2. □

We observe that in the corona product $G \odot H$, $\gamma_M(G \odot H)$ is determined by the structure of the graph H. In the following, we determine the metric-location-domination number of the corona product based on special properties of graph H.

Theorem 2.10. *Let G be any connected graph. Then $\gamma_M(G \odot H) = \gamma(G \odot H)$ holds if and only if $H = K_1$ or $H = K_2$.*

Proof. Let G be a connected graph m vertices. Consider the graph $G \odot H$. Let H_i be the i^{th} copy of H and h_{ij} be the j^{th} vertex adjacent to vertex g_i in $G \odot H$, for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, k$, where $k = |V(H)|$. First, assume that $H = K_1$ or $H = K_2$ and define $S = \{h_{11}, h_{21}, \ldots, h_{m1}\}$. We show that $\gamma_M(G \odot H) = \gamma(G \odot H)$ if $H = K_1$ or $H = K_2$.

Case 1.1: If $H = K_1$, then S contains all vertices of degree one and since all vertices in G adjacent to a vertex in S, S dominates all vertices in $V(G \odot K_1) - S$. Therefore, $\gamma(G \odot K_1) \leq m$. Next, suppose S' is a dominating set of $G \odot K_1$ with $|S'| = m - 1$. Then there are g_i and h_{i1}, for some i that are not in S'. This implies that h_{i1} is not dominated by S', a contradiction. Hence, $\gamma(G \odot K_1) = m$. Now, we also prove that S is a resolving set of $G \odot K_1$. This is clear since if $u = g_i$ and $v = g_j$, for some i and j with $i \neq j$, then $1 = d(u, h_{i1}) < d(v, h_{i1})$ and so $r(u|S) \neq r(v|S)$. Thus, $\gamma_M(G \odot K_1) \leq m$. Since S is a minimum dominating set of $G \odot K_1$, S is a minimum metric-locating-dominating set of $G \odot K_1$. Hence, $\gamma_M(G \odot K_1) = \gamma(G \odot K_1)$.

Case 1.2: If $H = K_2$, then h_{i1}, h_{i2} and g_i forms a triangle in $G \odot K_2$, for each i. Thus, $S = \{h_{11}, h_{21}, \ldots, h_{m1}\}$ is a dominating set of $G \odot K_2$. Therefore, $\gamma(G \odot K_2) \leq m$. Next, suppose S' is a dominating set of $G \odot K_2$ with $|S'| = m - 1$. Then there are h_{i1}, h_{i2} and g_i not in S', for some i. Thus the vertices h_{i1} and h_{i2} are not dominated by S', a contradiction. Hence, $\gamma(G \odot K_2) = m$. Now, we show that S is also a resolving set of $G \odot K_2$. Let u and v be any two distinct vertices of $(G \odot K_2) \backslash S$. If $u = g_i$ and $v = g_j$, for some i and j with $i \neq j$, then $1 = d(u, h_{i1}) < d(v, h_{i1})$ and so $r(u|S) \neq r(v|S)$. If $(u = g_i$ and $v = h_{i2})$ or $(u = g_i$ and $v = h_{j2})$, for some i and j with $i \neq j$, then clearly $r(u|S) \neq r(v|S)$. Thus S is a resolving set of

$G \odot K_2$. We get $\gamma_M(G \odot K_2) \leq m$. Since S is a minimum dominating set of $G \odot K_2$, S is a minimum metric-locating-dominating-set of $G \odot K_2$. Hence $\gamma_M(G \odot K_2) = m$. We conclude that $\gamma_M(G \odot H) = \gamma(G \odot H)$ if $H = K_1$ or $H = K_2$.

Conversely, if $\gamma_M(G \odot H) = \gamma(G \odot H)$, for any connected graph G of order m, then we show that $H = K_1$ or $H = K_2$. If H is a special graph, then, by Theorem 2.4, we have $\gamma_M(G \odot H) = \beta(G \odot H) + \gamma(G \odot H)$. Since $\beta(G \odot H) \geq 1$, $\gamma_M(G \odot H) = \beta(G \odot H) + \gamma(G \odot H) > \gamma(G \odot H)$, a contradiction. Therefore, H must not be a special graph. Then H can be complete or not. If H is a complete graph K_n, then $n = 1$ or $n = 2$ since otherwise $\gamma_M(G \odot H) > \gamma(G \odot H) = m$. If H is not complete, then there exist two distinct nonadjacent vertices h_{i1} and h_{i2} both are adjacent to g_i in $G \odot H$, for some i. Since $\gamma(G \odot H) = m$ and $\gamma_M(G \odot H) = \gamma(G \odot H)$, to dominate both vertices h_{i1} and h_{i2} g_i must be in a metric-locating-dominating set of G. But both vertices h_{i1} and h_{i2} will be not resolved, a contradiction. Therefore, there is no non-complete H satisfying $\gamma_M(G \odot H) = \gamma(G \odot H)$. Consequently, H must be either K_1 or K_2. □

Theorem 2.11. *Let G be a connected graph. Let H be a graph other than K_1 and K_2. Then $\gamma_M(G \odot H) = \beta(G \odot H)$ if and only if $\gamma_M(K_1 \odot H) = \beta(K_1 \odot H)$ and one of the following holds:*

(a) every minimum metric-locating-dominating set of $K_1 \odot H$ does not contain x, where x is a vertex adjacent to all vertices of H.

(b) there is a minimum metric-locating-dominating set L of $K_1 \odot H$ containing x and $L - \{x\}$ is a dominating set of $K_1 \odot H$, where x is a vertex adjacent to all vertices of H.

Proof. Let G be a connected graph. Let H be a graph other than K_1 and K_2. Let $\gamma_M(K_1 \odot H) = \beta(K_1 \odot H)$ and every minimum metric-locating-dominating set of $K_1 \odot H$ does not contain x, where x is a vertex adjacent to all vertices of H. Let A be a minimum metric-locating-dominating set of $K_1 \odot H$ does not contain x. Let A_i be the corresponding set A in the i^{th} copy of $K_1 \odot H$ in $G \odot H$, for each i. Then $M = A_1 \cup A_2 \cup \cdots \cup A_n$ is a minimum resolving set as well as a metric-locating-dominating set of $G \odot H$. Therefore, $\gamma_M(G \odot H) = \beta(G \odot H)$. If there is a minimum metric-locating-dominating set A of $K_1 \odot H$ containing x and $A - \{x\}$ is dominating, then $M = A_1 \cup A_2 \cup \cdots \cup A_n - \{x_1, x_2, \cdots, x_n\}$ is a minimum resolving set as well as a metric-locating-dominating set of $G \odot H$.

Conversely, if $\gamma_M(G \odot H) = \beta(G \odot H)$ holds for any connected graph G and any graph $H \neq K_s$ for $s = 1, 2$, then, by Theorems 2.4 and 2.10, the proof follows. □

Theorem 2.12. *For any connected graph G and any graph H, either $\gamma_M(G \odot H) = \max\{\gamma(G \odot H)$ or $\gamma_M(G \odot H) = \gamma(G \odot H) + \beta(G \odot H)$.*

Proof. The proof follows using Theorems 2.4, 2.10 and 2.11. □

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