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Hankel Determinant for a Class of Analytic Functions Involving the Q-Ruscheweyh Derivative and the Symmetric Differential Operator

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Abstract

By using the q-Ruscheweyh derivative and the symmetric differential operator $\Upsilon(q,\lambda,m)f(z)$ which is defined recently by the authors, a new set of the upper bounds for the second Hankel determinant are given.

1 Introduction

Let S denote the class of normalized analytic univalent functions f of the form

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots = z + \sum_{k=2}^{\infty} a_k z^k, \quad \text{where} \quad z \in U \tag{1.1}$$

which are analytic in the open unit disc $U = [z : |z| < 1]$.
In a recent paper [3], we introduced a new differential operator involving the q-Ruscheweyh derivative and the symmetric differential as follows:

$$\Upsilon_{(q,\lambda,\mu,m)}f(z) = z + \sum_{k=2}^{\infty} \frac{[k+\lambda-1]_q![k(\mu-(1-\mu)(-1)^k)]^m}{[\lambda]_q![k-1]_q!} a_k z^k, \tag{1.2}$$

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where $\lambda > \frac{1}{2}$,$0 \leq \mu \leq 1$ and m is an integer.

Here the q-Ruscheweyh derivative was motivated by Aldweby and Darus [2] and the symmetric differential was motivated by Ibrahim and Darus [8].

The study of q-calculus has attracted many researchers in the area of geometric function theory. Following Jackson [9] and Aral [4], we give some necessary definitions about q-calculus as follows

$$\partial_q f(z) = \frac{f(z) - f(zq)}{z(1-q)}, \quad (z \neq 0, \ z \in (0,1))$$

and

$$\int_0^z f(t)\partial_q t = z(1-q)\sum_{k=0}^{\infty} q^k f(zq^k), \quad q \in (0,1).$$

Now when $k = 1, 2, 3...$ and $z \in U$, we can see that

$$\partial_q \left\{ \sum_{k=1}^{\infty} a_k z^k \right\} = \sum_{k=1}^{\infty} [k]_q a_k z^{k-1}$$

where

$$[k]_q = 1 + \sum_{m=1}^{k-1} q^m = \frac{1-q^k}{1-q}, \quad [0]_q = 0.$$

For any non-negative integer k, the q-number shift factorial is defined by

$$[k]_q! = \begin{cases} 1 & k = 0 \\ [1]_q[2]_q[3]_q \dots [k]_q & k = 1, 2, 3, \dots . \end{cases}$$

Also, the q-generalized Pochhamer symbol for $x > 0$ is given as

$$[k, q]_k = \begin{cases} 1 & k = 0 \\ [x]_q[x+1] \dots [x+k-1]_q & k = 1, 2, 3, \dots . \end{cases}$$

In 1976, Noonan and Thomas [13] stated that the function f given by (1.1) and $q \in N = 1, 2, 3, ...$, the q^{th} Hankel determinant of f defined as

$$H_q(k) = \det \begin{pmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \dots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \dots & a_{n+2q-2} \end{pmatrix},$$

where the second Hankel determinant for $q = 2$ and $k = 2$ is defined by

$$H_2(2) = \det \begin{pmatrix} a_2 & a_3 \\ a_3 & a_4 \end{pmatrix} = \left| a_2 a_4 - a_3^2 \right|.$$

Studying this determinant has been the subject of many researchers such as Abubaker and Darus [1], Darus and Thomas [5] and Srivastava et al. [14] who have investigated a sharp upper limit for a second Hankel determinant. In 2020, Elhaddad and Darus [7] studied the second Hankel determinant of a class of analytic functions defined using the q-analogue of Ruscheweyh operator, which Aldweby and Darus [2] introduced.

In the present paper, we find the upper bound for the Hankel determinant for $q = 2$ and $k = 2$ by using the same technique introduced by Elhaddad and Darus [7] in the following class of univalent function that is defined using Alshammari and Darus (1.2).

Definition 1.1. *Let f be given by (1.1). Then $f \in R(q, \lambda, \mu, m)$ if it satisfies the inequality*

$$\Re\{(D_q(\Upsilon_{(q,\lambda,\mu,m)} f(z))\} > 0, \quad z \in U. \tag{1.3}$$

Note that, when $\lambda = 0$, $m = 0$ and $q \to 1^-$, the class $\Upsilon_{(q,\lambda,\mu,m)}$ is reduced to the class studied by MacGregor [12] and Janteng et al. [10]. Moreover, when $m = 0$ and $q \to 1^-$, the class $\Upsilon_{(q,\lambda,\mu,m)}$ reduces to the class studied by Elhaddad and Darus [7].

2 Preliminary Results

Let ϑ be the family of all functions ρ analytic in U for which $\Re\{\rho(z)\} > 0$

$$\rho(z) = 1 + c_1 z + c_2 z^2 + ... \tag{2.4}$$

for $z \in U$.

Lemma 2.1. *[6] If $\rho \in \vartheta$, then $|c_k| \leq 2$ for each $k \in N$ and the inequality is sharp.*

Lemma 2.2. *[11] Let the function $\rho \in \vartheta$ be given by the power series. Then*

$$2c_2 = c_1^2 + x(4 - c_1^2) \tag{2.5}$$

for some x, $|x| \leq 1$, and

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z \tag{2.6}$$

for some z, $|z| \leq 1$.

3 Main result

Theorem 3.1. *Let* $f \in R(q, \lambda, \mu, m)$. *Then*

$$\left|a_2 a_4 - a_3^2\right| \leq \frac{16([2]_q!)^2}{([\lambda+2]_q)^2([\lambda+1]_q)^2(3-6\mu)^{2m}}.$$

Proof. Since $f \in R(q, \lambda, \mu, m)$, it follows from (1.3) that

$$D_q[\Upsilon(q, \lambda, \mu, m) f(z)] = \rho(z). \tag{3.7}$$

Comparing the coefficients implies

$$a_2 = \frac{c_1}{\Gamma_2}, \tag{3.8}$$

$$a_3 = \frac{2c_2}{\Gamma_3}, \tag{3.9}$$

$$a_4 = \frac{3c_3}{\Gamma_4}, \tag{3.10}$$

where $\Gamma_2 = [2]_q[\lambda+1]_q[4\mu-2]^m$, $\Gamma_3 = \frac{[1]_q}{[2]_q!}[\lambda+2]_q[\lambda+1][3-6\mu]^m$, and $\Gamma_4 = \frac{[1]_q}{[3]_q!}[\lambda+3]_q[\lambda+2]_q[\lambda+1]_q[8\mu-4]^m$.

Now applying Lemmas 2.1 and 2.2 to equations (3.9) and (3.10) implies

$$a_3 = \frac{1}{\Gamma_3}[c_1^2 + x(4-c_1^2], \tag{3.11}$$

$$a_4 = \frac{3}{4\Gamma_4}[c_1^3 + 2(4-c_1^2)c_1 x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2)z]. \tag{3.12}$$

Thus,

$$|a_2 a_4 - a_3^2| \leq \frac{1}{4\Gamma_2\Gamma_3^2\Gamma_4}\left[\begin{array}{c} [3\Gamma_3^2 - 4\Gamma_2\Gamma_4]c^4 + [6\Gamma_3^2 - 8\Gamma_2\Gamma_4]c^2(4-c^2)\rho \\ +[4\Gamma_2\Gamma_4(4-c^2) - 3\Gamma_3^2c^2 - 6\Gamma_3^2 c](4-c^2)\rho^2 + 6\Gamma_3^2 c(4-c^2) \end{array}\right]$$

$= F(c, \rho)$,
where $0 \leq c \leq 2$ and $0 \leq \rho \leq 1$.

We maximize the function $F(c, \rho)$ on the closed square $[0, 2] \times [0, 1]$

$$\frac{\partial F}{\partial \rho} = \frac{1}{4\Gamma_2\Gamma_3^2\Gamma_4}\left[(6\Gamma_3^2 - 8\Gamma_2\Gamma_4)c^2(4-c^2) + (4\Gamma_2\Gamma_4(4-c^2) - 3\Gamma_3^2c^2 - 6\Gamma_3^2 c)2(4-c^2)\rho\right]$$

$$\frac{\partial^2 F}{\partial \rho} = \frac{1}{4\Gamma_2\Gamma_3^2\Gamma_4}\left[(4\Gamma_2\Gamma_4(4-c^2) - 3\Gamma_3^2c^2 - 6\Gamma_3^2c)2(4-c^2)\right] < 0.$$

We have $\frac{\partial F}{\partial \rho} < 0$ for $0 < c < 2$, and $0 < \rho < 1$. Thus, $F(c,\rho)$ can not have a maximum in the interior of the closed square $[0,2] \times [0,1]$. Moreover, for fixed $c \in [0,2]$, we have

$$\max_{0\le\rho\le1} F(c,\rho) = F(c,1) = G(c).$$

Then,

$$G(c) = \frac{1}{4\Gamma_2\Gamma_3^2\Gamma_4}\begin{bmatrix} [3\Gamma_3^2 - 4\Gamma_2\Gamma_4]c^4 + [6\Gamma_3^2 - 8\Gamma_2\Gamma_4]c^2(4-c^2) \\ +[4\Gamma_2\Gamma_4(4-c^2) - 3\Gamma_3^2c^2 - 6\Gamma_3^2c](4-c^2) + 6\Gamma_3^2c(4-c^2) \end{bmatrix}$$

and

$$G'(c) = \frac{1}{4\Gamma_2\Gamma_3^2\Gamma_4}\begin{bmatrix} [3\Gamma_3^2 - 4\Gamma_2\Gamma_4]4c^2 + [(12\Gamma_3^2 - 16\Gamma_2\Gamma_4) - 8\Gamma_2\Gamma_4 - 6\Gamma_3^2](4-c^2) \\ -2c^2(6\Gamma_3^2 - 8\Gamma_2\Gamma_4) - 2(4\Gamma_2\Gamma_4)(4-c^2) + 6c^2\Gamma_3^2 \end{bmatrix} c < 0.$$

So $G'(c) < 0$ for $0 < c < 2$ and has a real critical point at $c = 0$. Also, $G(c) > G(2)$. Therefore, $\max_{0\le c\le2} G(c)$ occurs at $c = 0$. As a result, the upper bound of $F(c,\rho)$ depends on $\rho = 1$ and $c = 0$.
Hence,

$$\left|a_2a_4 - a_3^2\right| \le \frac{16}{\Gamma_3^2}.$$

Consequently,

$$\left|a_2a_4 - a_3^2\right| \le \frac{16([2]_q!)^2}{([\lambda+2]_q)^2([\lambda+1]_q)^2(3-6\mu)^{2m}}. \tag{3.13}$$

4 Conclusion

The result here is basically on a Hankel determinant for a class of analytic functions involving the q-derivative of Ruscheweyh operator and symmetric differential operator given by Defintion 1.1. The results of Hankel determinant can also be studied for the class of starlike and convex functions involving both aforementioned operators.

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