International Journal of Mathematics and
Computer Science, **18**(2023), no. 1, 105–109

Weak (Λ, sp)-continuity and $\theta(\Lambda, sp)$-closed sets

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(Received August 29, 2022, Accepted October 4, 2022)

Abstract

The purpose of the present paper is to investigate some characterizations of upper and lower weakly (Λ, sp)-continuous multifunctions by utilizing $\theta(\Lambda, sp)$-closed sets.

1 Introduction

The notion of continuity is an important concept for the study in topological spaces. Weaker and stronger forms of open sets play an important role in the search of generalizations of continuity. Using weaker and stronger forms of open sets, many authors have introduced and investigated various types of generalizations of continuity for functions and multifunctions. Levine [5] introduced the concept of weakly continuous functions as a generalization of continuity. Popa [9] and Smithson [10] independently introduced the notion of weakly continuous multifunctions. The present authors introduced and studied other weak forms of continuous multifunctions: weakly α-continuous multifunctions [8], weakly β-continuous multifunctions [7]. Abd El-Monsef et al. [1] introduced a weak form of open sets called β-open sets. Noiri and Hatir [6] introduced the notion of Λ_{sp}-sets in terms of the concept of

Key words and phrases: $\theta(\Lambda, sp)$-open set, upper weakly (Λ, sp)-continuous multifunction, lower weakly (Λ, sp)-continuous multifunction.
AMS (MOS) Subject Classifications: 54C08, 54C60.
ISSN 1814-0432, 2023, http://ijmcs.future-in-tech.net

β-open sets and investigated the notion of Λ_{sp}-closed sets by using Λ_{sp}-sets. Boonpok [4] introduced the concepts of (Λ, sp)-open sets and (Λ, sp)-closed sets which are defined by utilizing the notions of Λ_{sp}-sets and β-closed sets. Moreover, Boonpok [4] studied some characterizations of upper and lower (Λ, sp)-continuous multifunctions. In this paper, we investigate some characterizations of upper and lower weakly (Λ, sp)-continuous multifunctions.

2 Preliminaries

Throughout this paper, unless explicitly stated, spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed. For a subset A of a topological space (X, τ), $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ represent the closure and the interior of A, respectively. A subset A of a topological space (X, τ) is said to be *β-open* [1] if $A \subseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$. The complement of a β-open set is said to be *β-closed*. The family of all β-open sets of a topological space (X, τ) is denoted by $\beta(X, \tau)$. A subset $\Lambda_{sp}(A)$ [6] is defined as follows:
$\Lambda_{sp}(A) = \cap\{U \mid A \subseteq U, U \in \beta(X, \tau)\}$. A subset A of a topological space (X, τ) is said to be a *Λ_{sp}-set* [6] if $A = \Lambda_{sp}(A)$. A subset A of a topological space (X, τ) is called *(Λ, sp)-closed* [4] if $A = T \cap C$, where T is a Λ_{sp}-set and C is a β-closed set. The complement of a (Λ, sp)-closed set is said to be *(Λ, sp)-open*. The family of all (Λ, sp)-open sets of a topological space (X, τ) is denoted by $\Lambda_{sp}O(X, \tau)$. Let A be a subset of a topological space (X, τ). A point $x \in X$ is called a *(Λ, sp)-cluster point* [4] of A if $A \cap U \neq \emptyset$ for every (Λ, sp)-open set U of X containing x. The set of all (Λ, sp)-cluster points of A is called the *(Λ, sp)-closure* [4] of A and is denoted by $A^{(\Lambda, sp)}$. The union of all (Λ, sp)-open sets contained in A is called the *(Λ, sp)-interior* [4] of A and is denoted by $A_{(\Lambda, sp)}$.

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and always assume that $F(x) \neq \emptyset$ for all $x \in X$. Following [2], for a multifunction $F : X \to Y$, we denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively; that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$.

3 Characterizations

In this section, we investigate some characterizations of upper and lower weakly (Λ, sp)-continuous multifunctions.

Definition 3.1. [3] *A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be:*

(i) upper weakly (Λ, sp)-continuous if, for each $x \in X$ and each (Λ, sp)-open set V of Y containing $F(x)$, there exists a (Λ, sp)-open set U of X containing x such that $F(U) \subseteq V^{(\Lambda,sp)}$;

(ii) lower weakly (Λ, sp)-continuous if, for each $x \in X$ and each (Λ, sp)-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a (Λ, sp)-open set U of X containing x such that $F(z) \cap V^{(\Lambda,sp)} \neq \emptyset$ for each $z \in U$.

Definition 3.2. [4] *Let A be a subset of a topological space (X, τ). The $\theta(\Lambda, sp)$-closure of A, $A^{\theta(\Lambda,sp)}$, is defined as follows:*

$$A^{\theta(\Lambda,sp)} = \{x \in X \mid A \cap \ U^{(\Lambda,sp)} \neq \emptyset \text{ for each } U \in \Lambda_{sp}O(X, \tau) \text{ containing } x\}.$$

A subset A of a topological space (X, τ) is said to be $\theta(\Lambda, sp)$-*closed* [4] if $A = A^{\theta(\Lambda,sp)}$. The complement of a $\theta(\Lambda, sp)$-closed set is said to be $\theta(\Lambda, sp)$-*open*. The union of all $\theta(\Lambda, sp)$-open sets contained in A is called the $\theta(\Lambda, sp)$-*interior* of A and is denoted by $A_{\theta(\Lambda,sp)}$.

Lemma 3.3. *If $F : (X, \tau) \to (Y, \sigma)$ is lower weakly (Λ, sp)-continuous, then for each $x \in X$ and each subset B of Y with $F(x) \cap B_{\theta(\Lambda,sp)} \neq \emptyset$, there exists $U \in \Lambda_{sp}O(X, \tau)$ containing x such that $U \subseteq F^{-}(B)$.*

Proof. Since $F(x) \cap B_{\theta(\Lambda,sp)} \neq \emptyset$, there exists a nonempty (Λ, sp)-open set V of Y such that $V^{(\Lambda,sp)} \subseteq B$ and $F(x) \cap V \neq \emptyset$. Since F is lower weakly (Λ, sp)-continuous, there exists a (Λ, sp)-open set U of X containing x such that $F(z) \cap V^{(\Lambda,sp)} \neq \emptyset$ for each $z \in U$ and hence $U \subseteq F^{-}(B)$. □

Theorem 3.4. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower weakly (Λ, sp)-continuous;

(2) $[F^{+}(B)]^{(\Lambda,sp)} \subseteq F^{+}(B^{\theta(\Lambda,sp)})$ for every subset B of Y;

(3) $F(A^{(\Lambda,sp)}) \subseteq [F(A)]^{\theta(\Lambda,sp)}$ for every subset A of X.

Proof. (1) ⇒ (2): Let B be any subset of Y. Suppose that $x \notin F^+(B^{\theta(\Lambda,sp)})$. Then, we have $x \in F^-(Y - B^{\theta(\Lambda,sp)}) = F^-([Y - B]_{\theta(\Lambda,sp)})$. By Lemma 3.3, there exists $U \in \Lambda_{sp}O(X,\tau)$ containing x such that $U \subseteq F^-(Y - B) = X - F^+(B)$. Thus, $U \cap F^+(B) = \emptyset$ and hence $x \notin [F^+(B)]^{(\Lambda,sp)}$. This shows that $[F^+(B)]^{(\Lambda,sp)} \subseteq F^+(B^{\theta(\Lambda,sp)})$.

(2) ⇒ (3): Let A be any subset of X. By (2), we have $A^{(\Lambda,sp)} \subseteq [F^+(F(A))]^{(\Lambda,sp)} \subseteq F^+([F(A)]^{\theta(\Lambda,sp)})$. Thus, $F(A^{(\Lambda,sp)}) \subseteq [F(A)]^{\theta(\Lambda,sp)}$.

(3) ⇒ (1): Let V be any (Λ, sp)-open set of Y. By Lemma 43 of [4], $V^{(\Lambda,sp)} = V^{\theta(\Lambda,sp)}$ and by (3), $F([F^+(V)]^{(\Lambda,sp)}) \subseteq [F(F^+(V))]^{\theta(\Lambda,sp)} \subseteq V^{\theta(\Lambda,sp)} = V^{(\Lambda,sp)}$. Thus, $[F^+(V)]^{(\Lambda,sp)} \subseteq F^+(V^{(\Lambda,sp)})$ and by Theorem 2 of [3], F is lower weakly (Λ, sp)-continuous. □

Theorem 3.5. *For a multifunction $F : (X,\tau) \to (Y,\sigma)$, the following properties are equivalent:*

(1) F is upper weakly (Λ, sp)-continuous;

(2) $[F^-([B^{\theta(\Lambda,sp)}]_{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq F^-(B^{\theta(\Lambda,sp)})$ for every subset B of Y;

(3) $[F^-([B^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq F^-(B^{\theta(\Lambda,sp)})$ for every subset B of Y.

Proof. (1) ⇒ (2): Let B be any subset of Y. By Lemma 43 of [4], we have $B^{\theta(\Lambda,sp)}$ is (Λ, sp)-closed in Y, by Theorem 1 of [3], $[F^-([B^{\theta(\Lambda,sp)}]_{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq F^-(B^{\theta(\Lambda,sp)})$.

(2) ⇒ (3): The proof is obvious.

(3) ⇒ (1): Let K be any $r(\Lambda, sp)$-closed set of Y. By Lemma 43 of [4], $[K_{(\Lambda,sp)}]^{\theta(\Lambda,sp)} = [K_{(\Lambda,sp)}]^{(\Lambda,sp)} = K$ and by (3), we have $[F^-(K_{(\Lambda,sp)})]^{(\Lambda,sp)} = [F^-([[K_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq F^-([K_{(\Lambda,sp)}]^{\theta(\Lambda,sp)}) = F^-([K_{(\Lambda,sp)}]^{(\Lambda,sp)}) = F^-(K)$. Thus, by Theorem 1 of [3], F is upper weakly (Λ, sp)-continuous. □

Theorem 3.6. *For a multifunction $F : (X,\tau) \to (Y,\sigma)$, the following properties are equivalent:*

(1) F is lower weakly (Λ, sp)-continuous;

(2) $[F^+([B^{\theta(\Lambda,sp)}]_{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq F^+(B^{\theta(\Lambda,sp)})$ for every subset B of Y;

(3) $[F^+([B^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq F^+(B^{\theta(\Lambda,sp)})$ for every subset B of Y.

Proof. The proof is similar to that of Theorem 3.5. □

Definition 3.7. [3] *A function $f : (X,\tau) \to (Y,\sigma)$ is said to be weakly (Λ, sp)-continuous if, for each $x \in X$ and each (Λ, sp)-open set V of Y containing $f(x)$, there exists a (Λ, sp)-open set U of X containing x such that $f(U) \subseteq V^{(\Lambda,sp)}$.*

Corollary 3.8. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) f is weakly (Λ, sp)-continuous;

(2) $[f^{-1}([B^{\theta(\Lambda,sp)}]_{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq f^{-1}(B^{\theta(\Lambda,sp)})$ for every subset B of Y;

(3) $[f^{-1}([B^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq f^{-1}(B^{\theta(\Lambda,sp)})$ for every subset B of Y.

Acknowledgment. This research project was financially supported by Mahasarakham University.

References

[1] M. E. Abd El-Monsef, S. N. El-Deeb, R. A. Mahmoud, β-open sets and β-continuous mappings, Bull. Fac. Assiut Univ., **12,** (1983), 77–90.

[2] C. Berge, Espaces topologiques fonctions multivoques, Dunod, Paris, 1959.

[3] C. Boonpok, P. Pue-on, Upper and lower weakly (Λ, sp)-continuous multifunctions (submitted).

[4] C. Boonpok, (Λ, sp)-closed sets and related topics in topological spaces, WSEAS Tran. Math., **19,** (2020), 312–322.

[5] N. Levine, A decomposition of continuity in topological spaces, Amer. Math. Monthly, **68,** (1961), 44–46.

[6] T. Noiri, E. Hatir, Λ_{sp}-sets and some weak separation axioms, Acta Math. Hungar., **103,** no. 3, (2004), 225–232.

[7] V. Popa, T. Noiri, On upper and lower weakly β-continuous multifunctions, Annales Univ. Sci. Budapest, **43,** (2000), 25–48.

[8] V. Popa, T. Noiri, On upper and lower almost α-continuous multifunctions, Demonstr. Math., **29,** (1996), 381–396.

[9] V. Popa, Weakly continuous multifunctions, Boll. Un. Mat. Ital., (5) **15 (A)** (1978), 379–388.

[10] R. E. Smithson, Almost and weak continuity for multifunctions, Bull. Calcutta Math. Soc., **70,** (1978), 383–390.