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Characterizations of (Λ, p)-R_1 topological spaces

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Abstract

This paper is concerned with the concept of (Λ, p)-R_1 topological spaces. Moreover, several characterizations of (Λ, p)-R_1 topological spaces are investigated.

1 Introduction

The concept of R_1 topological spaces was first introduced and studied by Davis [4]. In 2000, Caldas et al. [2] defined two new classes of topological spaces called pre-R_0 and pre-R_1 spaces in terms of concept of preopen sets and investigated some characterizations of such spaces. In 2005, Cammaroto and Noiri [3] introduced a weak separation axiom m-R_0 in m-spaces which are equivalent to generalized topological spaces due to Lugojan [6]. In 2006, Noiri [8] introduced the notion of m-R_1 spaces and investigated several characterizations of m-R_0 spaces and m-R_1 spaces. Mashhour et al. [7] introduced and studied the notion of preopen sets. Ganster et al. [5] introduced the notions of a pre-Λ-set and a pre-V-set in topological spaces and investigated the fundamental properties of pre-Λ-sets and pre-V-sets.

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Boonpok and Viriyapong [1] introduced the notions of (Λ, p)-open sets and (Λ, p)-closed sets which are defined by utilizing the notions of Λ_p-sets and preclosed sets. In this paper, we introduce the concept of (Λ, p)-R_1 topological spaces. Moreover, we discuss several characterizations of (Λ, p)-R_1 topological spaces.

2 Preliminaries

Throughout this paper and unless explicitly stated, a space (X, τ) (or simply X) denotes a topological space on which no separation axioms are assumed. Let A be a subset of a topological space (X, τ). The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A of a topological space (X, τ) is said to be *preopen* [7] if $A \subseteq \mathrm{Int}(\mathrm{Cl}(A))$. The complement of a preopen set is called *preclosed*. The family of all preopen sets of a topological space (X, τ) is denoted by $PO(X, \tau)$. A subset $\Lambda_p(A)$ [5] is defined as follows: $\Lambda_p(A) = \cap\{U \mid A \subseteq U, U \in PO(X, \tau)\}$. A subset A of a topological space (X, τ) is called a Λ_p*-set* [5] if $A = \Lambda_p(A)$. A subset A of a topological space (X, τ) is called (Λ, p)*-closed* [1] if $A = T \cap C$, where T is a Λ_p-set and C is a preclosed set. The complement of a (Λ, p)-closed set is called (Λ, p)*-open*. The family of all (Λ, p)-open (resp. (Λ, p)-closed) sets in a topological space (X, τ) is denoted by $\Lambda_pO(X, \tau)$ (resp. $\Lambda_pC(X, \tau)$). Let A be a subset of a topological space (X, τ). A point $x \in X$ is called a (Λ, p)*-cluster point* [1] of A if $A \cap U \neq \emptyset$ for every (Λ, p)-open set U of X containing x. The set of all (Λ, p)-cluster points of A is called the (Λ, p)*-closure* [1] of A and is denoted by $A^{(\Lambda,p)}$.

Definition 2.1. [1] *A topological space (X, τ) is called (Λ, p)-R_0 if, for each (Λ, p)-open set U and each $x \in U$, $\{x\}^{(\Lambda,p)} \subseteq U$.*

Lemma 2.2. [1] *A topological space (X, τ) is (Λ, sp)-R_0 if and only if, for any points x and y in X, $\{x\}^{(\Lambda,sp)} \neq \{y\}^{(\Lambda,sp)}$ implies $\{x\}^{(\Lambda,sp)} \cap \{y\}^{(\Lambda,sp)} = \emptyset$.*

Definition 2.3. [1] *Let A be a subset of a topological space (X, τ). A subset $\Lambda_{(\Lambda,p)}(A)$ is defined as follows: $\Lambda_{(\Lambda,p)}(A) = \cap\{U \mid A \subseteq U, U \in \Lambda_pO(X, \tau)\}$.*

Definition 2.4. [1] *Let (X, τ) be a topological space and $x \in X$. A subset $\langle x\rangle_p$ is defined as follows: $\langle x\rangle_p = \Lambda_{(\Lambda,p)}(\{x\}) \cap \{x\}^{(\Lambda,p)}$.*

Lemma 2.5. *A topological space (X, τ) is (Λ, p)-R_0 if and only if $\langle x\rangle_p = \{x\}^{(\Lambda,p)}$ for each $x \in X$.*

3 Some characterizations of (Λ, p)-R_1 topological spaces

We begin this section by introducing the notion of (Λ, p)-R_1 topological spaces.

Definition 3.1. *A topological space (X, τ) is said to be (Λ, p)-R_1 if, for each x, y in X with $\{x\}^{(\Lambda,p)} \neq \{y\}^{(\Lambda,p)}$, there exist disjoint (Λ, p)-open sets U and V such that $\{x\}^{(\Lambda,p)} \subseteq U$ and $\{y\}^{(\Lambda,p)} \subseteq V$.*

Theorem 3.2. *A topological space (X, τ) is (Λ, p)-R_1 if and only if, for any points x, y in X with $\{x\}^{(\Lambda,p)} \neq \{y\}^{(\Lambda,p)}$, there exist (Λ, p)-closed sets F and K such that $x \in F$, $y \notin F$, $y \in K$, $x \notin K$ and $X = F \cup K$.*

Proof. Let x and y be any points in X with $\{x\}^{(\Lambda,p)} \neq \{y\}^{(\Lambda,p)}$. Then, there exist disjoint $U, V \in \Lambda_p O(X, \tau)$ such that $\{x\}^{(\Lambda,p)} \subseteq U$ and $\{y\}^{(\Lambda,p)} \subseteq V$. Now, put $F = X - V$ and $K = X - U$. Then, F and K are (Λ, p)-closed sets of X such that $x \in F$, $y \notin F$, $y \in K$, $x \notin K$ and $X = F \cup K$.

Conversely, let x and y be any points in X such that $\{x\}^{(\Lambda,p)} \neq \{y\}^{(\Lambda,p)}$. Then, $\{x\}^{(\Lambda,p)} \cap \{y\}^{(\Lambda,p)} = \emptyset$. In fact, if $z \in \{x\}^{(\Lambda,p)} \cap \{y\}^{(\Lambda,p)}$, then $\{z\}^{(\Lambda,p)} \neq \{x\}^{(\Lambda,p)}$ or $\{z\}^{(\Lambda,p)} \neq \{y\}^{(\Lambda,p)}$. In case $\{z\}^{(\Lambda,p)} \neq \{x\}^{(\Lambda,p)}$, by the hypothesis, there exists a (Λ, p)-closed set F such that $x \in F$ and $z \notin F$. Then, we have $z \in \{x\}^{(\Lambda,p)} \subseteq F$. This contradicts that $z \notin F$. In case $\{z\}^{(\Lambda,p)} \neq \{y\}^{(\Lambda,p)}$, similarly, this leads to the contradiction. Thus, $\{x\}^{(\Lambda,p)} \cap \{y\}^{(\Lambda,p)} = \emptyset$, by Lemma 2.2, (X, τ) is (Λ, p)-R_0. By the hypothesis, there exist (Λ, p)-closed sets F and K such that $x \in F$, $y \notin F$, $y \in K$, $x \notin K$ and $X = F \cup K$. Put $U = X - K$ and $V = X - F$. Then, U and V are (Λ, p)-open sets of X such that $x \in U$ and $y \in V$. Since (X, τ) is (Λ, p)-R_0, we have $\{x\}^{(\Lambda,p)} \subseteq U$, $\{y\}^{(\Lambda,p)} \subseteq V$ and also $U \cap V = \emptyset$. This shows that (X, τ) is (Λ, p)-R_1. □

Lemma 3.3. *If a topological space (X, τ) is (Λ, p)-R_1, then (X, τ) is (Λ, p)-R_0.*

Proof. Let $U \in \Lambda_p O(X, \tau)$ and $x \in U$. If $y \notin U$, then $U \cap \{y\}^{(\Lambda,p)} = \emptyset$ and $x \notin \{y\}^{(\Lambda,p)}$. Therefore, $\{x\}^{(\Lambda,p)} \neq \{y\}^{(\Lambda,p)}$. Since (X, τ) is (Λ, p)-R_1, there exists $V \in \Lambda_p O(X, \tau)$ such that $\{y\}^{(\Lambda,p)} \subseteq V$ and $x \notin V$. Thus, $V \cap \{x\}^{(\Lambda,p)} = \emptyset$ and hence $y \notin \{x\}^{(\Lambda,p)}$. Therefore, $\{x\}^{(\Lambda,p)} \subseteq U$. This shows that (X, τ) is (Λ, p)-R_0. □

Definition 3.4. [1] *Let A be a subset of a topological space (X, τ). The $\theta(\Lambda, p)$-closure of A, $A^{\theta(\Lambda,p)}$, is defined as follows:*

$$A^{\theta(\Lambda,p)} = \{x \in X \mid A \cap U^{(\Lambda,p)} \neq \emptyset \text{ for each } U \in \Lambda_p O(X, \tau) \text{ containing } x\}.$$

Theorem 3.5. *A topological space* (X,τ) *is* (Λ,p)-R_1 *if and only if* $\langle x\rangle_p = \{x\}^{\theta(\Lambda,p)}$ *for each* $x\in X$.

Proof. Let (X,τ) be (Λ,p)-R_1. By Lemma 3.3, (X,τ) is (Λ,p)-R_0 and by Lemma 2.5, $\langle x\rangle_p = \{x\}^{(\Lambda,p)} \subseteq \{x\}^{\theta(\Lambda,p)}$ for each $x\in X$. Thus, $\langle x\rangle_p \subseteq \{x\}^{\theta(\Lambda,p)}$ for each $x\in X$. In order to show the opposite inclusion, suppose that $y\notin\langle x\rangle_p$. Then, $\langle x\rangle_p\neq\langle y\rangle_p$. Since (X,τ) is (Λ,p)-R_0, by Lemma 2.5, $\{x\}^{(\Lambda,p)}\neq\{y\}^{(\Lambda,p)}$. Since (X,τ) is (Λ,p)-R_1, there exist disjoint (Λ,p)-open sets U and V of X such that $\{x\}^{(\Lambda,p)}\subseteq U$ and $\{y\}^{(\Lambda,p)}\subseteq V$. Since $\{x\}\cap V^{(\Lambda,p)}\subseteq U\cap V^{(\Lambda,p)}=\emptyset$, $y\notin\{x\}^{\theta(\Lambda,p)}$. Thus, $\{x\}^{\theta(\Lambda,p)}\subseteq\langle x\rangle_p$ and hence $\{x\}^{\theta(\Lambda,p)}=\langle x\rangle_p$.

Conversely, suppose that $\{x\}^{\theta(\Lambda,p)}=\langle x\rangle_p$ for each $x\in X$. Then, $\langle x\rangle_p = \{x\}^{\theta(\Lambda,p)}\supseteq\{x\}^{(\Lambda,p)}\supseteq\langle x\rangle_p$ and $\langle x\rangle_p=\{x\}^{(\Lambda,p)}$ for each $x\in X$. By Lemma 2.5, (X,τ) is (Λ,p)-R_0. Suppose that $\{x\}^{(\Lambda,p)}\neq\{y\}^{(\Lambda,p)}$. Thus, by Lemma 2.2, $\{x\}^{(\Lambda,p)}\cap\{y\}^{(\Lambda,p)}=\emptyset$. By Lemma 2.5, $\langle x\rangle_p\cap\langle y\rangle_p=\emptyset$ and hence $\{x\}^{\theta(\Lambda,p)}\cap\{y\}^{\theta(\Lambda,p)}=\emptyset$. Since $y\notin\{x\}^{\theta(\Lambda,p)}$, there exists a (Λ,p)-open set U of X such that $y\in U\subseteq U^{(\Lambda,p)}\subseteq X-\{x\}$. Let $V=X-U^{(\Lambda,p)}$, then $x\in V\in\Lambda_pO(X,\tau)$. Since (X,τ) is (Λ,p)-R_0, $\{y\}^{(\Lambda,p)}\subseteq U$, $\{x\}^{(\Lambda,p)}\subseteq V$ and $U\cap V=\emptyset$. This shows that (X,τ) is (Λ,p)-R_1. □

Corollary 3.6. *A topological space* (X,τ) *is* (Λ,p)-R_1 *if and only if* $\{x\}^{(\Lambda,p)} = \{x\}^{\theta(\Lambda,p)}$ *for each* $x\in X$.

Proof. Let (X,τ) be (Λ,p)-R_1. By Theorem 3.5, we have $\{x\}^{(\Lambda,p)}\supseteq\langle x\rangle_p = \{x\}^{\theta(\Lambda,p)}\supseteq\{x\}^{(\Lambda,p)}$ and hence $\{x\}^{(\Lambda,p)}=\{x\}^{\theta(\Lambda,p)}$ for each $x\in X$.

Conversely, suppose that $\{x\}^{(\Lambda,p)}=\{x\}^{\theta(\Lambda,p)}$ for each $x\in X$. First, we show that (X,τ) is (Λ,p)-R_0. Let $U\in\Lambda_pO(X,\tau)$ and $x\in U$. Let $y\notin U$. Then, $U\cap\{y\}^{(\Lambda,p)}=U\cap\{y\}^{\theta(\Lambda,p)}=\emptyset$. Thus, $x\notin\{y\}^{\theta(\Lambda,p)}$. There exists $V\in\Lambda_pO(X,\tau)$ such that $x\in V$ and $y\notin V^{(\Lambda,p)}$. Since $\{x\}^{(\Lambda,p)}\subseteq V^{(\Lambda,p)}$, $y\notin\{x\}^{(\Lambda,p)}$. This shows that $\{x\}^{(\Lambda,p)}\subseteq U$ and hence (X,τ) is (Λ,p)-R_0. By Lemma 2.5, $\langle x\rangle_p=\{x\}^{(\Lambda,p)}=\{x\}^{\theta(\Lambda,p)}$ for each $x\in X$. Thus, by Theorem 3.5, (X,τ) is (Λ,p)-R_1. □

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