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A Nonlinear Transmission Model with Analytical Soliton Solution

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Abstract

We study a nonlinear transmission line from an analytical point of view. Moreover, we find the nonlinear differential equation that describes the behavior of the electrical system and we solve it analytically, finding a soliton-type solution. Furthermore, these results are of great interest to engineers, physicists, mathematicians, and researchers of nonlinear sciences and their applications

1 Introduction

The classical nonlinear oscillator is common in many branches of physics and has been used to solve problems such as: a simple pendulum of large amplitude [1], nonlinear oscillations of plasma [2, 3], nonlinear vibrations through a plate holding a concentrated mass, a beam placed on an axially oscillating mount, nonlinear electrical circuits [4] and a wide variety of other problems. Thus, it is clear that the study of the nonlinear classical oscillator is of great importance due to its many applications. That said, it is perhaps

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more interesting to study not just one nonlinear oscillator but a chain of coupled nonlinear oscillators. This model allows, for example, to study the open states of a DNA molecule [5], waves in multicomponent plasma [6], the dynamics of solitons through a Nonlinear Transmission Line (NLTL) [7] and many others.

A soliton is a solitary wave that propagates without deformation in a nonlinear medium. It was first observed by John Scott Russel (1808-1882), a Scottish engineer, while studying the propagation of water waves in a shallow channels. Russell's observations did not capture the attention of the public until the end of the 20th century when the field of nonlinear science and solitons began to flourish as a result of great technological advances and the development of computers that allowed numerical and symbolic calculations, since the dynamics of solitons, described by partial differential equations, is very complex to deal with [8].

Today, many researchers agree that optical solitons are especially important due to the rise of fiber optic communications and related technologies. This is reflected in the large investments that have been made over the last decade in high-capacity electro-optical transmission systems and all types of optical information processing that are closely linked to nonlinear science [9].

One of the possibilities to introduce the study of nonlinear soliton dynamics without resorting to fiber optic laboratories is to use nonlinear transmission lines which are easier to build and much cheaper. A lumped NLTL is built from commercial devices, usually low-cost, with nonlinear capacitances and/or nonlinear inductances. These have allowed a more detailed study of the physical phenomena and the adjustment of the nonlinear effects using simulation software [10]. The parameters of the soliton such as its order, velocity and width depend strongly on the amplitude of the system input, frequency, bias voltage and load resistance.

In this paper, we present the study of the fundamental properties of solitons, such as its generation, propagation, interaction and dependency of the critical parameters in a NLTL which is made of nonlinear capacitor cells, resistors and dispersive inductances. In the first section, we obtain the physical model of a nonlinear transmission line as a chain of coupled nonlinear oscillators and the nonlinear and continuous differential equation describing the whole system. In the second section, we study the nonlinear equation and its particular cases as well as the propagation of solitons through the NLTL. The third section corresponds to the analysis of the results obtained.

The results can be of great interest for engineers, physicists, mathematicians and, in general, researchers of nonlinear science and its applications.

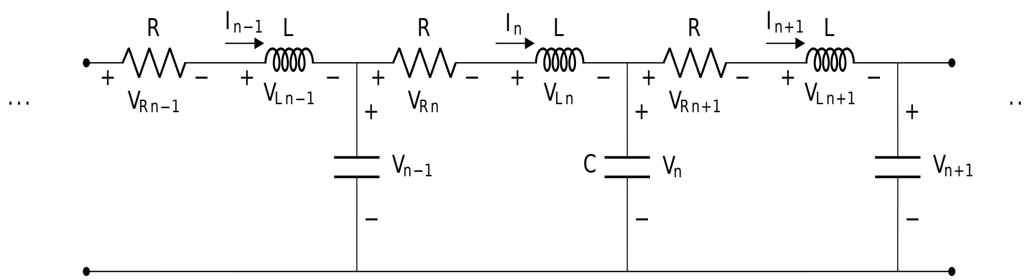

Figure 1: Circuit of a transmission line

2 Derivation of the model

Consider the following NLTL with n cells composed of inductances L, resistors R and nonlinear capacitors C. See Figure 1. Let $C = C_0(1 - bV_n)$ be a general way for describing the capacitance-charge dependency of the nonlinear capacitor. Thus, the current across the nth capacitor can be expressed as follows:

$$I_c = \frac{\mathrm{d}}{\mathrm{d}t}(CV_n) = \frac{\mathrm{d}C}{\mathrm{d}t}V_n + C\frac{\mathrm{d}V_n}{\mathrm{d}t} = C_0\frac{\mathrm{d}V_n}{\mathrm{d}t} - 2bC_0\frac{\mathrm{d}V_n}{\mathrm{d}t}V_n \tag{2.1}$$

Similarly, the nth inductance voltage is:

$$V_{Ln} = L\frac{\mathrm{d}I_n}{\mathrm{d}t} \tag{2.2}$$

Notice that the nth current is the same current across the nth resistor, leading to:

$$I_n = \frac{V_{Rn}}{R} \tag{2.3}$$

Kirchhoff's Laws state that $I_n = I_{n+1} + I_c$ and $V_{Rn} = V_{n-1} - V_{Ln} - V_n$. Replacing the equations 2.1, 2.2 and 2.3 in these two expressions we get:

$$\frac{V_{n-1} - V_{Ln} - V_n}{R} = \frac{V_n - V_{Ln+1} - V_{n+1}}{R} + I_c \tag{2.4}$$

$$\frac{V_{n-1}}{R} - \frac{L}{R}\frac{\mathrm{d}I_n}{\mathrm{d}t} - \frac{V_n}{R} = \frac{V_n}{R} - \frac{L}{R}\frac{\mathrm{d}I_{n+1}}{\mathrm{d}t} - \frac{V_{n+1}}{R} + C_0\frac{\mathrm{d}V_n}{\mathrm{d}t} - 2bC_0\frac{\mathrm{d}V_n}{\mathrm{d}t}V_n \tag{2.5}$$

Rearranging (2.5) gives:

$$C_0\frac{\mathrm{d}V_n}{\mathrm{d}t} - 2bC_0\frac{\mathrm{d}V_n}{\mathrm{d}t}V_n + \frac{L}{R}\frac{\mathrm{d}}{\mathrm{d}t}(I_n + I_{n+1}) = \frac{1}{R}(V_{n-1} + V_{n+1} - 2V_n) \tag{2.6}$$

The derivative of $I_n + I_{n+1}$ can be expressed as:

$$\frac{\mathrm{d}}{\mathrm{d}t}(I_n + I_{n+1}) = \frac{\mathrm{d}I_c}{\mathrm{d}t} = C_0\frac{\mathrm{d}^2V_n}{\mathrm{d}t^2} - 2bC_0\frac{\mathrm{d}^2V_n}{\mathrm{d}t^2}V_n - 2bC_0\left(\frac{\mathrm{d}V_n}{\mathrm{d}t}\right)^2 \tag{2.7}$$

Replacing the result (2.6) in the expression (2.5):

$$\begin{aligned} C_0\frac{\mathrm{d}V_n}{\mathrm{d}t} - 2bC_0\frac{\mathrm{d}V_n}{\mathrm{d}t}V_n + \frac{LC_0}{R}\frac{\mathrm{d}^2V_n}{\mathrm{d}t^2} - \frac{2bLC_0}{R}\frac{\mathrm{d}^2V_n}{\mathrm{d}t^2}V_n - \frac{2bLC_0}{R}\left(\frac{\mathrm{d}V_n}{\mathrm{d}t}\right)^2 \\ = \frac{1}{R}(V_{n-1} + V_{n+1} - 2V_n) \end{aligned} \tag{2.8}$$

Rearranging (2.8):

$$\begin{aligned} \frac{LC_0}{R}\frac{\mathrm{d}^2V_n}{\mathrm{d}t^2} - \frac{2bLC_0}{R}\left(\frac{\mathrm{d}V_n}{\mathrm{d}t}\right)^2 + C_0\frac{\mathrm{d}V_n}{\mathrm{d}t} - \frac{2bLC_0}{R}\frac{\mathrm{d}^2V_n}{\mathrm{d}t^2}V_n - 2bC_0\frac{\mathrm{d}V_n}{\mathrm{d}t}V_n \\ = \frac{1}{R}(V_{n-1} + V_{n+1} - 2V_n) \end{aligned} \tag{2.9}$$

Then Simplifying:

$$\begin{aligned} LC_0\frac{\mathrm{d}^2V_n}{\mathrm{d}t^2} - 2bLC_0\left(\frac{\mathrm{d}V_n}{\mathrm{d}t}\right)^2 + RC_0\frac{\mathrm{d}V_n}{\mathrm{d}t} - 2bLC_0\frac{\mathrm{d}^2V_n}{\mathrm{d}t^2}V_n - 2bRC_0\frac{\mathrm{d}V_n}{\mathrm{d}t}V_n \\ = V_{n-1} + V_{n+1} - 2V_n \end{aligned} \tag{2.10}$$

The excited wavelengths are much longer than the typical cell length ζ because the voltage varies slowly form one cell to the other; i.e.,

$$\frac{1}{V_0}\frac{\partial V}{\partial x} \ll \frac{1}{\zeta}.$$

Hence, the discrete lattice is taken to be continuous $V_n(t) \to V(x,\ t)$ and Taylor expansion is applied on nodal voltages as:

$$V_{n+1} \approx V(x,t) + \frac{\partial V(x,t)}{\partial x} + \frac{1}{2}\frac{\partial^2 V(x,t)}{\partial x^2} + \cdots \tag{2.11}$$

$$V_{n-1} \approx V(x,t) - \frac{\partial V(x,t)}{\partial x} + \frac{1}{2}\frac{\partial^2 V(x,t)}{\partial x^2} - \cdots \tag{2.12}$$

$$\frac{dV_n}{dt} = \frac{\partial V(x,t)}{\partial t} \qquad\qquad \frac{d^2 V_n}{dt^2} = \frac{\partial^2 V(x,t)}{\partial t^2} \tag{2.13}$$

Replacing (2.11, 2.12, 2.13) into (2.10,) we obtain the continuous equation; that is, a nonlinear differential equation that describes the behavior of the system.

$$\begin{gathered}\alpha\frac{\partial^2 V(x,t)}{\partial t^2} - \beta\left(\frac{\partial V(x,t)}{\partial t}\right)^2 + \gamma\frac{\partial V(x,t)}{\partial t} - \delta V(x,t)\frac{\partial^2 V(x,t)}{\partial t^2} \\ -\omega V(x,\ t)\frac{\partial V(x,t)}{\partial t} - \frac{\partial^2 V(x,t)}{\partial x^2} = 0\end{gathered} \tag{2.14}$$

where $\alpha = LC_0$, $\beta = 2bLC_0$, $\gamma = RC_0$, $\delta = 2bLC_0$, $\omega = 2bRC_0$

3 Anaytical Solutions

We look for a traveling wave solutions. Let $V(x,t) = u(x + \lambda t)$. Then

$$\begin{gathered}\alpha\frac{\partial^2 V(x,t)}{\partial t^2} - \beta\left(\frac{\partial V(x,t)}{\partial t}\right)^2 + \gamma\frac{\partial V(x,t)}{\partial t} - \delta V(x,t)\frac{\partial^2 V(x,t)}{\partial t^2} \\ -\omega V(x,t)\frac{\partial V(x,t)}{\partial t} - \frac{\partial^2 V(x,t)}{\partial x^2}\end{gathered} \tag{3.15}$$

$$= u''(\xi)\left(\alpha\lambda^2 - \delta\lambda^2 u(\xi) - 1\right) - \beta\lambda^2 u'(\xi)^2 + \lambda u'(\xi)(\gamma - \omega u(\xi))$$

3.1 First Case. $\delta = \beta = 0$

The ode to be solved reads

$$u''(\xi)\left(\alpha\lambda^2 - 1\right) + \lambda u'(\xi)\left[\gamma - \omega u(\xi)\right] = 0 \tag{3.16}$$

Assume a solution to the ode (2.10) in the form

$$u(\xi) = A + B\tanh(k\xi) \tag{3.17}$$

Then

$$u''(\xi)\left(\alpha\lambda^2-1\right)+\lambda u'(\xi)[\gamma-\omega u(\xi)]=Bk\,\mathrm{sech}^2(k\xi)\Big\{\lambda(\gamma-A\omega)\\ -\tanh(k\xi)\big[B\lambda\omega+2k(\alpha\lambda^2-1)\big]\Big\} \tag{3.18}$$

We choose the constants A and B so that

$$\begin{cases}\gamma-A\omega=0.\\ B\lambda\omega+2k\left(\alpha\lambda^2-1\right)=0.\end{cases} \tag{3.19}$$

Solving system (3.19) gives

$$A=\frac{\gamma}{\omega} \qquad B=-\frac{2k\left(\alpha\lambda^2-1\right)}{\lambda\omega} \tag{3.20}$$

3.2 Second Case. δ and $\beta \neq 0$

The ode in (3.15) does not admit a closed form solution. We seek a solution in the form

$$u(\xi)=\frac{\gamma}{\omega}-\frac{2k\left(\alpha\lambda^2-1\right)}{\lambda\omega}\tanh(k\xi)+\sum_{j=2}^{n}R_j\tanh^j(k\xi). \tag{3.21}$$

For the sake of simplicity, we take $n=3$. Plugging the identity intothe first four coefficients of tanh and sech and equating to zero gives the following algebraic system:

$$
\begin{gathered}
-8\alpha^2\delta k^3\lambda^4 + 16\alpha\delta k^3\lambda^2 - 8\delta k^3 + 16\alpha\delta k^2\lambda^3 R_3\omega - 16\delta k^2\lambda R_3\omega \\
-2\alpha k\lambda^2 R_2\omega^2 - 4\gamma\delta k\lambda^2 R_2\omega - 4\delta k\lambda^2 R_2^2\omega^2 - 6\delta k\lambda^2 R_3^2\omega^2 \\
+2kR_2\omega^2 + 5\lambda R_2 R_3\omega^3 = 0 \\[1em]
4\alpha\gamma\delta k^2\lambda^3 - 4\gamma\delta k^2\lambda + 12\alpha\delta k^2\lambda^3 R_2\omega - 12\delta k^2\lambda R_2\omega - 2\alpha k\lambda^2 R_3\omega^2 \\
-6\gamma\delta k\lambda^2 R_3\omega - 10\delta k\lambda^2 R_2 R_3\omega^2 + 2kR_3\omega^2 + 2\lambda R_2^2\omega^3 + 3\lambda R_3^2\omega^3 = 0 \\[1em]
4\alpha^2\beta k^3\lambda^4 + 8\alpha^2\delta k^3\lambda^4 - 8\alpha\beta k^3\lambda^2 - 16\alpha\delta k^3\lambda^2 + 4\beta k^3 + 8\delta k^3 \\
-12\alpha\beta k^2\lambda^3 R_3\omega - 44\alpha\delta k^2\lambda^3 R_3\omega + 12\beta k^2\lambda R_3\omega + 44\delta k^2\lambda R_3\omega \\
+4\beta k\lambda^2 R_2^2\omega^2 + 9\beta k\lambda^2 R_3^2\omega^2 + 6\gamma\delta k\lambda^2 R_2\omega + 10\delta k\lambda^2 R_2^2\omega^2 \\
+24\delta k\lambda^2 R_3^2\omega^2 - 10\lambda R_2 R_3\omega^3 = 0 \\[1em]
-2\Big[4\alpha\beta k^2\lambda^3 R_2 + 8\alpha\delta k^2\lambda^3 R_2 - 4\beta k^2\lambda R_2 - 8\delta k^2\lambda R_2 \\
+2\alpha k\lambda^2 R_3\omega - 6\beta k\lambda^2 R_2 R_3\omega - 6\gamma\delta k\lambda^2 R_3 - 14\delta k\lambda^2 R_2 R_3\omega \\
-2kR_3\omega + \lambda R_2^2\omega^2 + 3\lambda R_3^2\omega^2\Big] = 0
\end{gathered}
\tag{3.22}
$$

A solution is

$$
\begin{aligned}
R_2 &= \frac{2\left(4\alpha\delta k^2\lambda^3 - 4\delta k^2\lambda + \alpha k\lambda^2\omega - 3\gamma\delta k\lambda^2 - k\omega\right)}{5\lambda\omega(2\delta k\lambda - \omega)} \\
R_3 &= \frac{2\left(6\alpha\delta k^2\lambda^3 - 6\delta k^2\lambda - \alpha k\lambda^2\omega - 2\gamma\delta k\lambda^2 + k\omega\right)}{5\lambda\omega(2\delta k\lambda - \omega)}
\end{aligned}
\tag{3.23}
$$

We determine the constant k from the following sextics:

$$
\begin{aligned}
&841\omega^3\big[-336\alpha^3\beta^2\omega^3-656\alpha^3\beta\delta\omega^3-349\alpha^3\delta^2\omega^3+1872\alpha^2\beta^2\gamma\delta\omega^2\\
&+3416\alpha^2\beta\gamma\delta^2\omega^2+1605\alpha^2\gamma\delta^3\omega^2-3456\alpha\beta^2\gamma^2\delta^2\omega-6256\alpha\beta\gamma^2\delta^3\omega\\
&-2784\alpha\gamma^2\delta^4\omega+2112\beta^2\gamma^3\delta^3+4288\beta\gamma^3\delta^4+2176\gamma^3\delta^5\big]\\
&+16\omega^2\big[5376\alpha^2\beta^4\omega^2-196480\alpha^2\beta^3\delta\omega^2-355168\alpha^2\beta^2\delta^2\omega^2\\
&-130360\alpha^2\beta\delta^3\omega^2+45021\alpha^2\delta^4\omega^2-21504\alpha\beta^4\gamma\delta\omega+819968\alpha\beta^3\gamma\delta^2\omega\\
&+1785920\alpha\beta^2\gamma\delta^3\omega+1316272\alpha\beta\gamma\delta^4\omega+371136\alpha\gamma\delta^5\omega\\
&+21504\beta^4\gamma^2\delta^2-861184\beta^3\gamma^2\delta^3-1978048\beta^2\gamma^2\delta^4\\
&-1289280\beta\gamma^2\delta^5-193920\gamma^2\delta^6\big]k^2+8192\delta^2\omega\big[-1344\alpha\beta^4\omega\\
&+3760\alpha\beta^3\delta\omega+18124\alpha\beta^2\delta^2\omega+20407\alpha\beta\delta^3\omega+7329\alpha\delta^4\omega\\
&+2688\beta^4\gamma\delta-3712\beta^3\gamma\delta^2-25600\beta^2\gamma\delta^3-29544\beta\gamma\delta^4-10344\gamma\delta^5\big]k^4\\
&+1048576\delta^4(\beta+\delta)^2\left(336\beta^2+656\beta\delta+349\delta^2\right)k^6=0
\end{aligned}
\tag{3.24}
$$

In the case of a transmission line with $L = 2.7\,\mu H$, $R = 7\,\Omega$ and a commercial varicap BB 112 with $C_0 = 552\,\mathrm{pF}$, $b = 0.167$ (approximate values from the diode capacitance curve extracted from its datasheet) and taking into account that $\delta = \beta$, and $\lambda = 1/\sqrt{\beta}$ the sixth-order equation for k leads to:

$$
\begin{aligned}
5.202\times10^{-123}-4\times10^{-120}k^2-\\
5.642\times10^{-117}k^4+2.12\times10^{-113}k^6=0
\end{aligned}
\tag{3.25}
$$

From (3.24) we get that $k = 0.0231$. Considering (3.23) and the value of k in the equation (3.21), we obtain the following function:

$$
\begin{aligned}
V(x,t) = 2.563\tanh^3[0.0231(4.482\times10^7t+x)]\\
+3.046\tanh^2[0.023(4.482\times10^7t+x)]\\
-1.596\tanh[0.023(4.482\times10^7t+x)]+2.994
\end{aligned}
\tag{3.26}
$$

The graph of $V(x,t)$ is shown in figures 2 and 3.

Also, the circuit of a lumped NLTL from figure 4, made of varicaps as nonlinear capacitances with the same parameters as the last example is simulated using Multisim, the transmission line consists of 100 cells with a load on 100 kΩ. We measured the voltage on the varicaps of cells 20, 40, and 60, these results can be seen in figure 5.

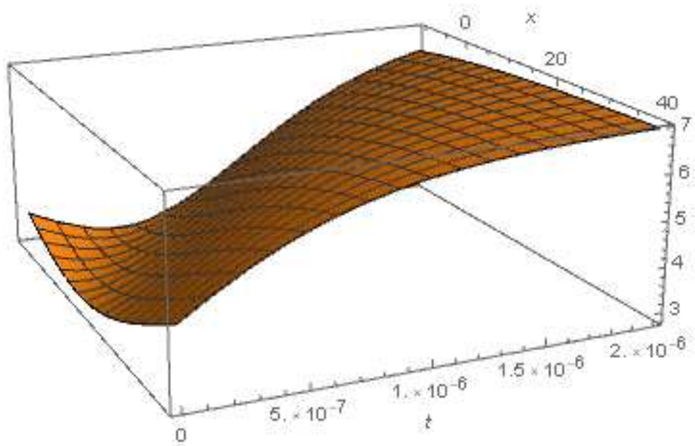

Figure 2: $V(x,t)$

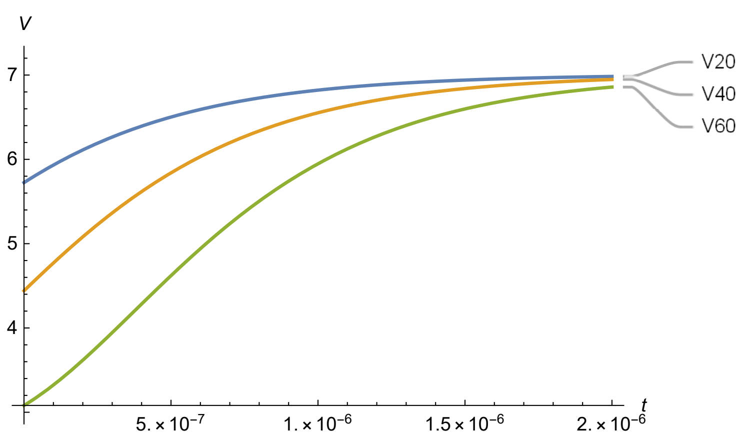

Figure 3: $V(x=20;40;60,t)$

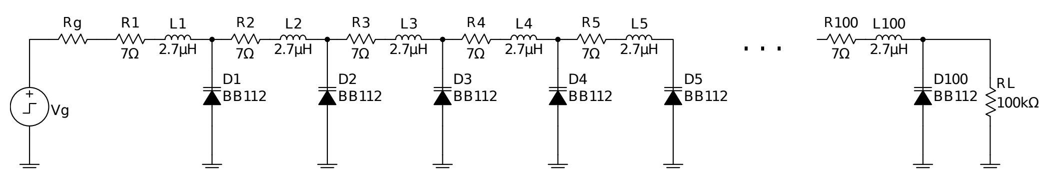

Figure 4: Lumped NLTL of 100 cells.

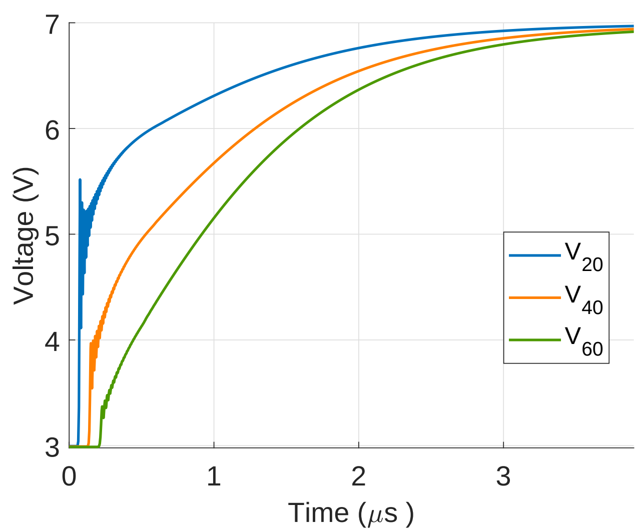

Figure 5: Voltage of each twenty cells using simulation software.

4 Conclusions

Nonlinear transmission lines provide an effective way to study nonlinear dispersive media leading to a way to precisely model electrical solitons. We studied the propagation of a solitonic wave through a NLTL with nonlinear capacitors analytically and obtained the results by means of simulation showing a qualitative agreement with the mathematical model. Finally, we suggest that more precise parameterizations of the capacitance nonlinearity in this type of NLTL, such as a higher degree polynomial approximation of the charge-dependent capacitance, could yield further useful results.

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