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A Note On New Near Algebraic Structures

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Abstract

In this paper, we study algebraic structures called modified near algebras, normed modified algebras obtained from new near structures called modified near modules.

1 Introduction

Magill, Jr. [1] introduced semilinear maps of the Euclidean spaces $\mathbb{R}^n$ into $\mathbb{R}$ as continuous maps satisfying $f(f(x)y) = f(x)f(y)$. Semilinear maps are used by removing the continuity as and when necessary and retain the above equality in other works [4], [5], [6]. Near algebras and normed near algebras were studied in [3]. In this paper, we use the notion of semilinear maps and obtain new near algebraic structures from algebraic structures called modified nearmodules [6].
Let $(M, +)$ be a group and let N be a near ring and suppose '$\cdot$' is a mapping of $N \times M$ into M.

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Definition 1.1. $(M, +, \cdot)$ *is called a* **near module** *over* N *if*

1. $(n_1 + n_2)m = n_1m + n_2m$ *for all* $n_1, n_2 \in N$ *and* $m \in M$*;*
2. $(n_1n_2)m = n_1(n_2m)$ *for all* $n_1, n_2 \in N$ *and* $m \in M$*.*

Remark 1.2. *Clearly, our near module is the* N***-group*** *introduced by Pilz.*

Definition 1.3. $(M, +, \cdot)$ *is called a* **modified near module***[6] over* N *if*

1. $n(m_1 + m_2) = nm_1 + nm_2$ *for all* $n \in N$ *and* $m_1, m_2 \in M$*;*
2. $(n_1n_2)m = n_1(n_2m)$ *for all* $n_1, n_2 \in N$ *and* $m \in M$*.*

Definition 1.4. *A* **near algebra** *[2,3] A is a linear space over* $\mathbb{R}$ *on which a multiplication is defined such that*

1. *A forms a semigroup under multiplication;*
2. *multiplication is right distributive with respect to addition;*
3. $\alpha(xy) = (\alpha x)y$ *for all* $x, y \in A$ *and* $\alpha \in \mathbb{R}$*.*

2 Main results

Definition 2.1. *A* **modified near algebra** M *is a modified near module* $(M, +, \cdot)$ *over a near ring* $(N, +, \cdot)$ *on which multiplication* $*$ *is defined such that*

1. $(M, *)$ *is a semigroup;*
2. $(m_1 + m_2) * m_3 = m_1 * m_3 + m_2 * m_3$*;*
3. $n(m_1 * m_2) = (nm_1) * m_2$ *for all* $m_1, m_2, m_3 \in M$ *and* $n \in N$*.*

Example 2.2. *Let* $(M, +, \cdot)$ *be a modified near module over a near ring* $(N, +, \cdot)$ *and let* $\widehat{M} = \{f/f : M \to M\}$*.*
Define $+, \cdot$ *and* $\circ$ *on* $\widehat{M}$ *as follows:*
$(f + g)(m) = f(m) + f(m), (nf)(m) = nf(m)$*,*
$(f \circ g)(m) = f(g(m))$ *for all* $f, g \in \widehat{M}, m \in M$ *and* $n \in N$*. Then* $(\widehat{M}, +, \cdot, \circ)$ *is a modified near algebra.*

The proof of the following theorem is straight forward.

Theorem 2.3. *Let $(M,+,\cdot)$ be a modified near module over a near ring $(N,+,\cdot)$, and let $Hom_N(M,M)=\{f/f:M\rightarrow M$ is a module homomorphism$\}$. Define $(f\oplus g)(m)=f(m)+g(m)$ and*
$(n\odot f)(m)=nf(m)$ for all $f,g\in Hom_N(M,M)$ and $n\in N$. Then $(Hom_N(M,M),\oplus,\odot)$ is a modified near module over $(N,+,\cdot)$.

Theorem 2.4. *Let $(M,+,\cdot)$ be a modified near module over a near ring $(N,+,\cdot)$ and $\Phi:M\rightarrow Hom_N(M,M)$ be a mapping such that $\Phi_{\Phi_{m_1}(m_2)}=\Phi_{m_1}\circ\Phi_{m_2}$ for all $m_1,m_2\in M$. Define $m_1*m_2=\Phi_{m_2}(m_1)$. Then $(M,+,\cdot,*)$ is a modified near algebra. Conversely, if $(M,+,\cdot,*)$ is a modified near algebra, define $\Phi:M\rightarrow Hom_N(M,M)$ by $\Phi_{m_1}(m_2)=m_2*m_1$ for all $m_1,m_2\in M$. Then $\Phi_{\Phi_{m_1}(m_2)}=\Phi_{m_1}\circ\Phi_{m_2}$. Also, $(M,+,\cdot,*)$ is a modified algebra if and only if $\Phi_{m_1+m_2}=\Phi_{m_1}+\Phi_{m_2}$ for all $m_1,m_2\in M$. Here, we denote $\Phi(m)$ by Φ_m.*

Proof. $(\Rightarrow)$ Define $m_1*m_2=\Phi_{m_2}(m_1)$ for all $m_1,m_2\in M$.
For any $m_1,m_2,m_3\in M$ and $n\in N$,
$m_1*(m_2*m_3)=m_1*(\Phi_{m_3}(m_2))=\Phi_{\Phi_{m_3}(m_2)}(m_1)=(\Phi_{m_3}\circ\Phi_{m_2})(m_1)=\Phi_{m_3}(\Phi_{m_2}(m_1))=\Phi_{m_3}(m_1*m_2)=(m_1*m_2)*m_3$.
Moreover, $(m_1+m_2)*m_3=\Phi_{m_3}(m_1+m_2)$
$=\Phi_{m_3}(m_1)+\Phi_{m_3}(m_2)$ [since $\Phi_{m_3}\in Hom_N(M,M)$]
$=m_1*m_3+m_2*m_3$.
Furthermore, $nm_1*m_2=\Phi_{m_2}(nm_1)$
$=n\Phi_{m_2}(m_1)$ [since $\Phi_{m_2}\in Hom_N(M,M)$]
$=n(m_1*m_2)$. Therefore, $(M,+,\cdot,*)$ is a modified near algebra.
$(\Leftarrow)$ First, we show that $\Phi_{m_1}\in Hom_N(M,M)$.
For any $m_2,m_3\in M$,
$\Phi_{m_1}(m_2+m_3)=(m_2+m_3)*m_1=m_2*m_1+m_3*m_1=\Phi_{m_1}(m_2)+\Phi_{m_1}(m_3)$
and $\Phi_{m_1}(nm_2)=(nm_2)*m_1=n(m_2*m_1)=n\Phi_{m_1}(m_2)$.
Therefore, $\Phi_m\in Hom_N(M,M)$ for all $m\in M$.
Now, we show that $\Phi_{\Phi_{m_1}(m_2)}=\Phi_{m_1}\circ\Phi_{m_2}$.
For any $m_3\in M$,
$\Phi_{\Phi_{m_1}(m_2)}(m_3)=m_3*[\Phi_{m_1}(m_2)]=m_3*(m_2*m_1)=(m_3*m_2)*m_1=\Phi_{m_1}(m_3*m_2)=\Phi_{m_1}(\Phi_{m_2}(m_3))=(\Phi_{m_1}\circ\Phi_{m_2})(m_3)$
$\Rightarrow\Phi_{\Phi_{m_1}(m_2)}=\Phi_{m_1}\circ\Phi_{m_2}$.
Suppose $(M,+,\cdot,*)$ is a modified algebra.
For any $m\in M$,
$\Phi_{m_1+m_2}(m)=m*(m_1+m_2)=m*m_1+m*m_2=\Phi_{m_1}(m)+\Phi_{m_2}(m)=(\Phi_{m_1}+\Phi_{m_2})(m)$
$\Rightarrow\Phi_{m_1+m_2}=\Phi_{m_1}+\Phi_{m_2}$.

Conversely, suppose $\Phi_{m_1+m_2} = \Phi_{m_1} + \Phi_{m_2}$.
For any $m_1, m_2, m_3 \in M$ and $n \in N$,
$m_1 * (m_2 + m_3) = \Phi_{m_2+m_3}(m_1) = (\Phi_{m_2} + \Phi_{m_3})(m_1) = \Phi_{m_2}(m_1) + \Phi_{m_3}(m_1)$
$= m_1 * m_2 + m_1 * m_3$.
Therefore, $(M, +, \cdot, *)$ is a modified algebra. □

By introducing a norm [2] on modified near modules, we have the following.

Definition 2.5. *A* **normed modified near module** *is a modified near module M over the field of reals on which there is defined a norm; i.e., a function which assigns to each element m in the space a real number $||m||$ such that*

1. $||m|| \geq 0$ *and* $||m|| = 0$ *if and only if* $m = 0$;
2. $||m_1 + m_2|| \leq ||m_1|| + ||m_2||$;
3. $||\alpha m|| = |\alpha|\ ||m||$ *for all* $m, m_1, m_2 \in M$ *and* $\alpha \in \mathbb{R}$.

Definition 2.6. *A* **normed modified near algebra** *is a modified near algebra $(M, +, \cdot, *)$ together with a norm with respect to which M is a normed modified near module such that $||m_1 * m_2|| \leq ||m_1||\ ||m_2||$ for all $m_1, m_2 \in M$.*

Definition 2.7. *Let M be a normed modified near module. A mapping $f : M \to \mathbb{R}$ is said to be a* **semilinear map** *if for every m_1, m_2 in M, $f(f(m_2)m_1) = f(m_2)f(m_1)$.*

Definition 2.8. *Let M be a normed modified near module. A mapping $\psi : M \to B(M)$ is said to be a* **semilinear operator**

1. Ψ *is continuous and*
2. $\Psi(\Psi(m_1)m_2) = \Psi(m_1) \circ \Psi(m_2)$ *for all* m_1, m_2 *in* M.

Theorem 2.9. *Let M be a normed modified near module and let $\Psi : M \to B(M)$ be a semilinear operator on M such that $\|\Psi(m)\| \leq \|m\|$ for every $m \in M$. Define $m_1 * m_2 = \Psi(m_2)m_1$ for all $m_1, m_2 \in M$. Then $(M, +, \cdot, *)$ is a normed modified near algebra. Conversely, if $(M, +, \cdot, *)$ is a normed modified near algebra, then $\Psi : M \to B(M)$ defined by $\Psi(m_2)m_1 = m_1 * m_2$ for all $m_1, m_2 \in M$ is a semilinear operator on M satisfying $\|\Psi(m)\| \leq \|m\|$ for all $m \in M$.*

Proof. By Theorem [2.4] $(M, +, \cdot, *)$ is a modified near algebra.
For any $m_1, m_2 \in M$, $\|m_1 * m_2\| = \|\Psi(m_2)m_1\| \leq \|\Psi(m_2)\| \, \|m_1\| \leq \|m_2\| \, \|m_1\|$.
So $(M, +, \cdot, *)$ is a normed modified near algebra.
Conversely, if $(M, +, \cdot, *)$ is a normed near algebra, define $\Psi(m) : M \to M$ by $\Psi(m)m_1 = m_1 * m$ for all m, m_1 in M. Then
$\Psi(m)(m_1 + m_2) = (m_1 + m_2) * m = m_1 * m + m_2 * m = \Psi(m)m_1 + \Psi(m)m_2$.
Moreover, $\Psi(m)(\alpha m_1) = (\alpha m_1) * m = \alpha(m_1 * m) = \alpha(\Psi(m)m_1)$.
By Theorem [2.4], Ψ is a semilinear operator on M.
Furthermore, $\|\Psi(m)m_1\| = \|m_1 * m\| \leq \|m_1\| \, \|m\|$.
Hence $\|\Psi(m)\| \leq \|m\|$. □

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