c-Convex Subgroups of Finite Dimensional Cyclically Ordered Free Abelian Groups

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Abstract

We consider a class of finite dimensional free abelian groups and prove that there is a cyclic order for them. Moreover, we give a full description of its c-convex subgroups; i.e., the counterpart of the notion of order ideals of totally ordered abelian groups.

1 Introduction

Adji and Raeburn [1] investigated ideal structure of Toeplitz algebras $\mathcal{T}(\Gamma)$ of totally ordered abelian groups $\Gamma$. Suppose $(\Gamma, +, \leq)$ is a totally ordered abelian group with the positive cone of $\Gamma$ is denoted by $\Gamma^+$. An order ideal of $\Gamma$ is a subgroup $I$ of $\Gamma$ which is order preserving; i.e, if $x \in \Gamma^+$ and $y \in I^+$ with $x \leq y$, then $x \in I$. The set of order ideals in $\Gamma$ is totally ordered under inclusion and it is denoted by $\Sigma(\Gamma)$. It was proved that the primitive ideals of $\mathcal{T}(\Gamma)$ were parametrized by disjoint union of the duals $\hat{I}$ of the order ideals $I$. The standard hull-kernel topology in the primitive ideal space of $\mathcal{T}(\Gamma)$ was then identified by the corresponding topology in the disjoint union of the duals $\hat{I}$ defined in [1, Definition 4.1].

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The notion of Toeplitz algebra of cyclically ordered abelian group was constructed in [12, 13]. A cyclically ordered (abelian) group is an (abelian) group \((G, +)\) equipped with a ternary relation on \(R\) satisfying some conditions. The class of cyclically ordered group is larger than the class of linearly ordered group. Any linearly ordered (abelian) group \((G, +, \leq)\) can be converted into a cyclically ordered (abelian) group \((G, +, R)\), but some cyclically ordered (abelian) group can not be reversed to its linear counterpart; for example, we can define a cyclic order \(R\) on the modulo group \(Z_n\) of integers, but there is no linear order can be induced from \(R\). Therefore, the notion of Toeplitz algebra of cyclically ordered abelian group is a generalization of the Toeplitz algebra of totally ordered abelian groups considered in [1].

Each irreducible representation factors through an irreducible representation of \(T(\Gamma / I)\) for some \(I \in \Sigma(\Gamma)\) which enables the primitive ideals to be parametrized by the disjoint union \(X := \bigsqcup \{\hat{I} : I \in \Sigma(\Gamma)\}\) [1, Theorem 3.1]. As a result, a crucial ingredient [1] is the set \(\Sigma(\Gamma)\) of order ideals in \(\Gamma\) which is totally ordered under inclusion. Many attempts to generalize the results in [1] rely on the class of \(\Sigma(\Gamma)\). Some generalization of results of [1] can be found in [2, 9, 11, 10].

Since the role of \(\Sigma(\Gamma)\) is very crucial, it will be very helpful in analysis when we have a clear description about each order ideal \(I\). In this article, it will be considered a class of finite dimensional free abelian groups. This class is very important because there is a cyclic order for this class. The order is a generalization of the linear order considered in [1]. The positive cones of cyclically ordered free abelian groups are semigroups, and this is a very crucial part in our analysis. Then we discuss some properties of \(c\)-convex subgroups, the counterpart of the notion of order ideals of totally ordered abelian groups. The notion of \(c\)-convex subgroups itself has appeared in many references including [6].

In Sections 2 and 3, we discuss the notion and basic properties of cyclically ordered groups and \(c\)-convex subgroups. Then, in Section 4, we give concrete examples of cyclically ordered abelian groups and description of their \(c\)-convex subgroups. These examples become fundamental aspects of our analysis on finite dimensional free abelian groups. Our results should serve as a basis for possible generalizations of the results in [1, 2, 9, 11, 10, 13] for Toeplitz algebras of cyclically ordered abelian groups.
2 Preliminaries on Cyclically Ordered Groups

2.1 Cyclic Order on a Set and Cyclically Ordered Groups

A usual order which is basically a binary relation can not be imposed on a circle, because it is not in accordance with orientation on a circle. Motivated by geometric problems, Huntington [4] introduced a ternary relation (back then was called a triadic relation) which is sufficient to define an orientation of a circle. This relation extends in full generality which is sufficient to define an order on any set, and it is compatible with an orientation on a circle. Such an order is called a cyclic order.

**Definition 2.1.** Let $S$ be any non empty set. A ternary relation $R$ on $S$ is any subset $R$ of $S^3$, written as $R(x,y,z)$, to indicate $(x,y,z) \in R$. A cyclic order on $S$ is a ternary relation $R$ on $S$ satisfying the following axioms:

- **R1**: if $R(x,y,z)$, then $x \neq y \neq z \neq x$ (R is strict),
- **R2**: if $x, y, z \in G$ such that $x \neq y \neq z \neq x$, then $R(x,y,z)$ or $R(x,z,y)$ (R is total),
- **R3**: if $R(x,y,z)$, then $R(y,z,x)$ (R is cyclic),
- **R4**: if $R(x,y,z)$ and $R(y,u,z)$, then $R(x,u,z)$ (R is transitive).

The set $S$ is called a cyclically ordered set.

Rieger [7] introduced the notion of cyclically ordered group which generalized the well established theory of ordered group. Basically, a cyclically ordered group is a group equipped with a cyclic order which satisfies this additional axiom which assures its compatibility with the group operation:

- **R5**: if $R(x,y,z)$, then $R(u + x + v, u + y + v, u + z + v)$ (R is compatible with the operation on $G$).

Therefore, formally we have the following definition:

**Definition 2.2.** A cyclically ordered group $(G, +, R)$ is a group $(G, +)$ which is equipped with a ternary relation $R$ on $G$ that satisfies the axioms R1 through R5.

**Remark 2.3.** Suppose $(L, +, <)$ is a linearly ordered group. A cyclic order $R_L$ on $L$ can be induced from the linear order $<$ on $L$ by defining:

$$ R_L(x,y,z) \iff x < y < z \text{ or } y < z < x \text{ or } z < x < y. $$

The group $L$ is then cyclically ordered. Therefore, every linearly ordered group is basically a cyclically ordered group by the induced cyclic order.
2.2 c-Convex Subgroups

If \((G, +, \leq)\) is a linearly ordered group, then one way to construct a quotient group is through the order ideal; i.e., a normal subgroup which preserves the order; for example, see [1, 2, 8, 9, 10, 11].

We want to discuss a parallel approach for cyclically ordered groups. Let \((G, +, R)\) be a cyclically ordered group. In [6], there was a discussion on how to construct its quotient; i.e., through special normal subgroups called c-convex subgroups.

**Definition 2.4.** [6] Suppose \((G, +, R)\) is a cyclically ordered group. A subgroup \(H\) of \(G\) is called a c-convex if one of the following conditions is fulfilled:

(i) \(H = G\);

(ii) For every non zero \(h \in H\), we have \(2h \neq 0\). If \(h \in H\), \(g \in G\), \(R(-h, 0, h), R(-h, g, h)\), then \(g \in H\).

**Definition 2.5.** Suppose \((G, +, R)\) is a cyclically ordered group. The family of all normal c-convex subgroups of \(G\) is denoted by \(\Sigma(G)\).

In line with the set of all order ideals of a linearly ordered group is partially ordered by the set-theoretical inclusion, the set \(\Sigma(G)\) of all normal c-convex subgroups of a cyclically ordered group \(G\) is linearly ordered [6, Lemma 3.3].

3 Main Results

3.1 Direct Sum of Groups of Integers

Suppose \(J = \{1, \ldots, n\}\) and consider the direct sum \(G = \oplus_{j \in J} G_j\), where \(G_j = \mathbb{Z}\), \(\forall j \in J\). The lexicographic order on \(G\) is defined by

\[
(x_j) \leq_{\text{lex}} (y_j) \iff x_1 < y_1 \text{ or: } \exists n_0 \in J \text{ such that } x_j = y_j \forall j < n_0 \text{ and } x_{n_0} \leq y_{n_0}.
\]

This is a linear order on \(G\) and it turns out that the group is a totally ordered abelian group and is denoted by \(G = \bigoplus_{\text{lex } j \in J} G_j\). We then induce a cyclic order \(R\) from the linear order \(\leq_{\text{lex}}\) as follows:

\[
R(g_1, g_2, g_3) \iff g_1 <_{\text{lex}} g_2 <_{\text{lex}} g_3 \text{ or } g_2 <_{\text{lex}} g_3 <_{\text{lex}} g_1 \text{ or } g_3 <_{\text{lex}} g_1 <_{\text{lex}} g_2
\] (3.1)
for every element \( g_1, g_2, g_3 \in G \) with \( g_1 \neq g_2 \neq g_3 \neq g_1 \). The group \( G \) is now cyclically ordered and every element is comparable (complete order). We denote that as \( G = \bigoplus_{cogj \in J} G_j \).

We are going to figure out how do \( c \)-convex subgroups of \( G = \bigoplus_{cogj \in J} G_j \) for \( G_j = \mathbb{Z} \) look like. Recall that every subgroup of \( G \) is normal. Hence all \( c \)-convex subgroups of \( G \) are elements of \( \Sigma(G) \).

**Theorem 3.1.** Suppose \( J = \{1, ..., n\} \), \( G = \bigoplus_{cogj \in J} G_j \) with \( G_j = \mathbb{Z} \) for \( i \in J \).

Then the set \( I_i := \{(x_j) | x_j = 0 \text{ for } j < i\} \) is a \( c \)-convex subgroup of \( G \).

**Proof.** It is clear that \( I_i \) is a subgroup of \( G \) and \( 2x \neq 0 \), for every nonzero element \( x \in I_i \). Suppose \((g_j) \in G \) and \((x_j) \in I_i \) such that

\[
R(-(x_j), 0_G, (x_j)), \tag{3.2}
\]

and

\[
R(-(x_j), (g_j), (x_j)), \tag{3.3}
\]

we claim that \((g_j) \in I_i \).

By definition of cyclic order, the condition (3.2) is equivalent to three cases. A series of messy computations shows that only case

\[ -(x_j) <_{lex} 0_G <_{lex} (x_j) \]

needs to be considered, because the remaining two cases never happen. For example, the case \( 0_G <_{lex} (x_j) <_{lex} (-x_j) \) is equivalent to

\[ 0 < x_1 \text{ or } \exists n_0 \in J \ni 0 = x_j \forall j < n_0 \text{ and } x_{n_0} > 0, \]

and

\[ x_1 < -x_1 \text{ or } \exists m_0 \in J \ni x_j = -x_j \forall j < m_0 \text{ and } x_{m_0} < -x_{m_0} \]

which is impossible to occur. Similar computation shows that for the condition (3.3), we only need to consider the case

\[ -(x_j) <_{lex} (g_j) <_{lex} (x_j). \tag{3.4} \]

By definition, (3.4) is equivalent to

\[
-x_1 < g_1 \quad \text{or} \quad \exists n_0 \in J \ni -x_j = g_j \forall j < n_0 \text{ and } -x_{n_0} < g_{n_0} \tag{A, B}
\]
and
\[ g_1 < x_1 \] in label
\[ (A \lor B) \land (C \lor D) = (A \land C) \lor (A \land D) \lor (B \land C) \lor (B \land D). \]

A messy computation shows that only \((B \land D)\) need to be considered, as the other three are impossible to occur. Furthermore, there is only a single possible subcase that needs to be considered; i.e., when \(n_0, m_0 \geq i\). This implies \(g_j = x_j = 0 \forall j < \min\{n_0, m_0\}\). Consequently, \((g_j) \in I_i\).

If \(H\) is a subgroup of \(G = \bigoplus_{cogj \in J} G_j\) with \(G_j = \mathbb{Z}\) for \(i \in J\), Theorem II.1.6 of [3] implies that \(H = \bigoplus_{cogj \in J} g_jG_j\) with \(g_j \in \mathbb{Z}\).

**Lemma 3.2.** If \(H = \bigoplus_{cogj \in J} g_jG_j\) is a non trivial \(c\)-convex subgroup of \(G = \bigoplus_{cogj \in J} G_j\) with \(G_j = \mathbb{Z}\) for \(i \in J\) such that \(H \subseteq I_k\) for some \(k \in J\), then there is \(k_0 \in J\) with \(k_0 \geq k\) such that \(g_j = 0\) for all \(j < k_0\), and \(g_j = 1\) for all \(j \geq k_0\).

**Proof.** From the definition of \(I_k\), it is clear that \(g_j = 0\) for all \(j < k\). Suppose \(k_0 \geq k\) in \(J\) such that \(g_{k_0}\) is the first non zero number. We claim that \(g_j = 1\) for all \(j \geq k_0\). Suppose on the contrary that there is \(j_0 \in J\) with \(j_0 \geq k_0\) such that \(g_{j_0} \neq 1\). Suppose \(h \in H\) such that
\[
\begin{align*}
    h(j) &= \left\{
    \begin{array}{ll}
    g_{j_0} & \text{if } j = j_0 \\
    0 & \text{otherwise,}
    \end{array}
    \right.
\end{align*}
\]
and \(g \in G\) such that
\[
\begin{align*}
    g(j) &= \left\{
    \begin{array}{ll}
    1 & \text{if } j = j_0 \\
    0 & \text{otherwise.}
    \end{array}
    \right.
\end{align*}
\]
Then we get \(R(\neg h, 0, h)\) and \(R(\neg h, g, h)\). But \(g \not\in H\). This contradicts the hypothesis that \(H\) is a \(c\)-convex subgroup. Hence our assumption is not correct, and the proof is complete.

**Theorem 3.3.** Suppose \(J = \{1, ..., n\}\), \(G = \bigoplus_{cogj \in J} G_j\) with \(G_j = \mathbb{Z}\), \(\forall j \in J\).

If \(H = \bigoplus_{cogj \in J} g_jG_j\) is a non trivial \(c\)-convex subgroup of \(G\), then \(H = I_i\) for some \(i \in J\).
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Proof. We claim that there is \( k \in J \) such that \( H \subseteq I_k \). Lemma 3.2 implies that there is \( k_0 \geq k \) such that \( H = I_{k_0} \) and the proof is complete. Suppose on the contrary that there is no \( k \in J \) such that \( H \subseteq I_k \). Since \( \Sigma(G) \) is totally ordered by inclusion, \( I_k \subseteq H \) for all \( k \in J \). This implies that \( g_1 \neq 0 \) and \( g_j = 1 \) for all \( j > 1 \). Suppose \( h \in H \) such that

\[
h(j) = \begin{cases} g_i & \text{if } j = 0 \\ 0 & \text{otherwise,} \end{cases}
\]

and \( g \in G \) such that

\[
g(j) = \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{otherwise.} \end{cases}
\]

Then \( R(-h,0,h) \) and \( R(-h,g,h) \). But \( g \not\in H \). This contradicts the hypothesis that \( H \) is a \( c \)-convex subgroup of \( G \).

3.2 Finite Dimensional Free Abelian Groups

Suppose \( G \) is a finite dimensional free abelian group. A well known theorem in group theory (for example Theorem 1.1 of [3]) says that \( G \) is isomorphic to a finite direct sum of copies of additive group \( \mathbb{Z} \) of integers. Recall that if \( X = \{x_1, ..., x_n\} \) is a basis for \( G \), then every element of \( G \) can be written as \( \sum_{i=1}^{n} k_i x_i \) with \( k_i \in \mathbb{Z} \). The following mapping is then an isomorphism between \( G \) and \( \bigoplus_{i=1}^{n} G_i \) with \( G_i = \mathbb{Z} \):

\[
\psi : \sum_{i=1}^{n} k_i x_i \mapsto (k_i x_i)_{i=1}^{n}. \tag{3.5}
\]

Recall that the cyclic order \( R \) for \( \bigoplus_{i=1}^{n} G_i \) is given by (3.1). We induce a cyclic order \( R' \) for \( G \) by

\[
R'(g,h,j) \iff R(\psi(g),\psi(h),\psi(j)), \tag{3.6}
\]

and then \( G \) is cyclically ordered. Therefore, we get the following theorem.

Theorem 3.4. If \( G \) is an \( n \)-dimensional free abelian group, then \( G \) is cyclically ordered under the cyclic order given by (3.6); i.e., (3.5) gives an isomorphism between \( G \) and \( \bigoplus_{i=1}^{n} \mathbb{Z} \) as cyclically ordered groups.

If \( G \) is a free abelian group of finite dimension, then Theorem 3.4 implies that \( G \) is cyclically ordered. Examples of \( c \)-convex subgroups of \( G \) are given by Corollary 3.5, and finally a full description of \( \Sigma(G) \) is given by Corollary 3.6.
Corollary 3.5. Suppose $G$ is a finite dimensional free abelian group with basis $X = \{x_1, ..., x_n\}$ and $i \in \{1, ..., n\}$. Then $J_i := \{\sum_{j \geq i} \lambda_j x_j | \lambda_j \in \mathbb{Z}, x_j \in X\}$ is a $c$-convex subgroup of $G$.

Proof. From Theorem 3.4, $\psi : \sum_{i=1}^{n} k_i x_i \mapsto (k_i x_i)_{i=1}^{n}$ gives an isomorphism between $G$ and $\bigoplus_{i=1}^{n} \mathbb{Z}$. Hence each $J_i$ in $G$ is nothing but $I_i$ in $\bigoplus_{i=1}^{n} \mathbb{Z}$. Therefore, $J_i$ is a $c$-convex subgroup of $G$ by Theorem 3.1.

Corollary 3.6. Suppose $G$ is a finite dimensional free abelian group with basis $X = \{x_1, ..., x_n\}$. If $H$ is a non-trivial $c$-convex subgroup of $G$, then $H = J_i := \{\sum_{j \geq i} \lambda_j x_j | \lambda_j \in \mathbb{Z}, x_j \in X\}$ for some $i \in \{1, ..., n\}$. Therefore, $\Sigma(G)$ consists of $J_i$ for all $i \in \{1, ..., n\}$.

Proof. From Theorem 3.4, $\psi : \sum_{i=1}^{n} k_i x_i \mapsto (k_i x_i)_{i=1}^{n}$ gives an isomorphism between $G$ and $\bigoplus_{i=1}^{n} \mathbb{Z}$. If $H$ is a $c$-convex subgroup of $G$, then $\psi(H)$ is a $c$-convex subgroup of $\bigoplus_{i=1}^{n} \mathbb{Z}$. Hence, from Theorem 3.3, $\psi(H) = I_i := \{(x_j)|x_j = 0 \text{ for } j < i\}$ for some $i \in \{1, ..., n\}$, therefore

$$H = \psi^{-1}(\{(x_j)|x_j = 0 \text{ for } j < i\}) = \{\sum_{j \geq i} \lambda_j x_j | \lambda_j \in \mathbb{C}, x_j \in X\}.$$ 

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