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c-Convex Subgroups of Finite Dimensional Cyclically Ordered Free Abelian Groups

Rizky Rosjanuardi, Sumanang Muhtar Gozali, Imam Nugraha Albania

Department of Mathematics Education
Faculty of Mathematics and Natural Sciences Education
Universitas Pendidikan Indonesia
Bandung, Indonesia

email: rizky@upi.edu

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Abstract

We consider a class of finite dimensional free abelian groups and prove that there is a cyclic order for them. Moreover, we give a full description of its c-convex subgroups; i.e., the counterpart of the notion of order ideals of totally ordered abelian groups.

1 Introduction

Adji and Raeburn [1] investigated ideal structure of Toeplitz algebras $\mathcal{T}(\Gamma)$ of totally ordered abelian groups Γ. Suppose $(\Gamma, +, \leq)$ is a totally ordered abelian group with the positive cone of Γ is denoted by Γ^+. An order ideal of Γ is a subgroup I of Γ which is order preserving; i.e, if $x \in \Gamma^+$ and $y \in I^+$ with $x \leq y$, then $x \in I$. The set of order ideals in Γ is totally ordered under inclusion and it is denoted by $\Sigma(\Gamma)$. It was proved that the primitive ideals of $\mathcal{T}(\Gamma)$ were parametrized by disjoint union of the duals $\hat{I}$ of the order ideals I. The standard hull-kernel topology in the primitive ideal space of $\mathcal{T}(\Gamma)$ was then identified by the corresponding topology in the disjoint union of the duals $\hat{I}$ defined in [1, Definition 4.1].

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The notion of Toeplitz algebra of cyclically ordered abelian group was constructed in [12, 13]. A cyclically ordered (abelian) group is an (abelian) group $(G, +)$ equipped with a ternary relation on R satisfying some conditions. The class of cyclically ordered group is larger than the class of linearly ordered group. Any linearly ordered (abelian) group $(G, +, \leq)$ can be converted into a cyclically ordered (abelian) group $(G, +, R)$, but some cyclically ordered (abelian) group can not be reversed to its linear counterpart; for example, we can define a cyclic order R on the modulo group $\mathbb{Z}_n$ of integers, but there is no linear order can be induced from R. Therefore, the notion of Toeplitz algebra of cyclically ordered abelian group is a generalization of the Toeplitz algebra of totally ordered abelian groups considered in [1].

Each irreducible representation factors through an irreducible representation of $\mathcal{T}(\Gamma/I)$ for some $I \in \Sigma(\Gamma)$ which enables the primitive ideals to be parametrized by the disjoint union $X := \bigsqcup\{\hat{I} : I \in \Sigma(\Gamma)\}$ [1, Theorem 3.1]. As a result, a crucial ingredient [1] is the set $\Sigma(\Gamma)$ of order ideals in Γ which is totally ordered under inclusion. Many attempts to generalize the results in [1] rely on the class of $\Sigma(\Gamma)$. Some generalization of results of [1] can be found in [2, 9, 11, 10].

Since the role of $\Sigma(\Gamma)$ is very crucial, it will be very helpful in analysis when we have a clear description about each order ideal I. In this article, it will be considered a class of finite dimensional free abelian groups. This class is very important because there is a cyclic order for this class. The order is a generalization of the linear order considered in [1]. The positive cones of cyclically ordered free abelian groups are semigroups, and this is a very crucial part in our analysis. Then we discuss some properties of c-convex subgroups, the counterpart of the notion of order ideals of totally ordered abelian groups. The notion of c-convex subgroups itself has appeared in many references including [6].
In Sections 2 and 3, we discuss the notion and basic properties of cyclically ordered groups and c-convex subgroups. Then, in Section 4, we give concrete examples of cyclically ordered abelian groups and description of their c-convex subgroups. These examples become fundamental aspects of our analysis on finite dimensional free abelian groups. Our results should serve as a basis for possible generalizations of the results in [1, 2, 9, 11, 10, 13] for Toeplitz algebras of cyclically ordered abelian groups.

2 Preliminaries on Cyclically Ordered Groups

2.1 Cyclic Order on a Set and Cyclically Odered Groups

A usual order which is basically a binary relation can not be imposed on a circle, because it is not in accordance with orientation on a circle. Motivated by geometric problems, Hutington [4] introduced a ternary relation (back then was called a triadic relation) which is sufficient to define an orientation of a circle. This relation extends in full generality which is sufficient to define an order on any set, and it is compatible with an orientation on a circle. Such an order is called a cyclic order.

Definition 2.1. *Let S be any non empty set. A ternary relation R on S is any subset R of S^3, written as $R(x, y, z)$, to indicate $(x, y, z) \in R$. A cyclic order on S is a ternary relation R on S satisfying the following axioms:*

R1 : if $R(x, y, z)$, then $x \neq y \neq z \neq x$ (R is strict),

R2 : if $x, y, z \in G$ such that $x \neq y \neq z \neq x$, then $R(x, y, z)$ or $R(x, z, y)$ (R is total),

R3 : if $R(x, y, z)$, then $R(y, z, x)$ (R is cyclic),

R4 : if $R(x, y, z)$ and $R(y, u, z)$, then $R(x, u, z)$ (R is transitive). The set S is called a cyclically ordered set.

Rieger [7] introduced the notion of cyclically ordered group which generalized the well established theory of ordered group. Basically, a cyclically ordered group is a group equipped with a cyclic order which satisfies this additional axiom which assures its compatibility with the group operation:

R5 : if $R(x, y, z)$, then $R(u + x + v, u + y + v, u + z + v)$ (R is compatible with the operation on G).

Therefore, formally we have the following definition:

Definition 2.2. *A cyclically ordered group $(G, +, R)$ is a group $(G, +)$ which is equipped with a ternary relation R on G that satisfies the axioms $R1$ through $R5$.*

Remark 2.3. *Suppose $(L, +, <)$ is a linearly ordered group. A cyclic order R_L on L can be induced from the linear order $<$ on L by defining:*

$$R_L(x, y, z) \Leftrightarrow x < y < z \text{ or } y < z < x \text{ or } z < x < y.$$

The group L is then cyclically ordered. Therefore, every linearly ordered group is basically a cyclically ordered group by the induced cyclic order.

2.2 c-Convex Subgroups

If $(G, +, \leq)$ is a linearly ordered group, then one way to construct a quotient group is through the order ideal; i.e., a normal subgroup which preserves the order; for example, see [1, 2, 8, 9, 10, 11].
We want to discuss a parallel approach for cyclically ordered groups. Let $(G, +, R)$ be a cyclically ordered group. In [6], there was a discussion on how to construct its quotient; i.e., through special normal subgroups called c-convex subgroups.

Definition 2.4. *[6] Suppose $(G, +, R)$ is a cyclically ordered group. A subgroup H of G is called a c-convex if one of the following conditions is fulfilled:*

(i) $H = G$;

(ii) *For every non zero $h \in H$, we have $2h \neq 0$. If $h \in H$, $g \in G$, $R(-h, 0, h)$, $R(-h, g, h)$, then $g \in H$.*

Definition 2.5. *Suppose $(G, +, R)$ is a cyclically ordered group. The family of all normal c-convex subgroups of G is denoted by $\Sigma(G)$.*

In line with the set of all order ideals of a linearly ordered group is partially ordered by the set-theoretical inclusion, the set $\Sigma(G)$ of all normal c-convex subgroups of a cyclically ordered group G is linearly ordered [6, Lemma 3.3].

3 Main Results

3.1 Direct Sum of Groups of Integers

Suppose $J = \{1, ..., n\}$ and consider the direct sum $G = \oplus_{j\in J} G_j$, where $G_j = \mathbb{Z}$, $\forall j \in J$. The lexicographic order on G is defined by

$$(x_j) \leq_{lex} (y_j) \iff$$
$$x_1 < y_1 \text{ or: } \exists n_0 \in J \text{ such that } x_j = y_j \ \forall j < n_0 \text{ and } x_{n_0} \leq y_{n_0}.$$

This is a linear order on G and it turns out that the group is a totally ordered abelian group and is denoted by $G = \underset{lex j\in J}{\oplus} G_j$. We then induce a cyclic order R from the linear order $\leq_{lex}$ as follows:

$$R(g_1, g_2, g_3) \iff$$
$$g_1 <_{lex} g_2 <_{lex} g_3 \text{ or } g_2 <_{lex} g_3 <_{lex} g_1 \text{ or } g_3 <_{lex} g_1 <_{lex} g_2 \tag{3.1}$$

for every element $g_1, g_2, g_3 \in G$ with $g_1 \neq g_2 \neq g_3 \neq g_1$. The group G is now cyclically ordered and every element is comparable (complete order). We denote that as $G = \bigoplus_{cog_{j\in J}} G_j$.

We are going to figure out how do c-convex subgroups of $G = \oplus_{j\in J} G_j$ for $G_j = \mathbb{Z}$ look like. Recall that every subgroup of G is normal. Hence all c-convex subgroups of G are elements of $\Sigma(G)$.

Theorem 3.1. *Suppose $J = \{1, ..., n\}$, $G = \bigoplus_{cog_{j\in J}} G_j$ with $G_j = \mathbb{Z}$ for $i \in J$. Then the set $I_i := \{(x_j) | x_j = 0$ for $j < i\}$ is a c-convex subgroup of G.*

Proof. It is clear that I_i is a subgroup of G and $2x \neq 0$, for every nonzero element $x \in I_i$. Suppose $(g_j) \in G$ and $(x_j) \in I_i$ such that

$$R(-(x_j), 0_G, (x_j)), \tag{3.2}$$

and

$$R(-(x_j), (g_j), (x_j)), \tag{3.3}$$

we claim that $(g_j) \in I_i$.

By definition of cyclic order, the condition (3.2) is equivalent to three cases. A series of messy computations shows that only case

$$-(x_j) <_{lex} 0_G <_{lex} (x_j)$$

needs to be considered, because the remaining two cases never happen. For example, the case $0_G <_{lex} (x_j) <_{lex} (-x_j)$ is equivalent to

$$0 < x_1 \text{ or } \exists n_0 \in J \ni 0 = x_j \ \forall j < n_0 \text{ and } x_{n_0} > 0,$$

and

$$x_1 < -x_1 \text{ or } \exists m_0 \in J \ni x_j = -x_j \ \forall j < m_0 \text{ and } x_{m_0} < -x_{m_0}$$

which is impossible to occur. Similar computation shows that for the condition (3.3), we only need to consider the case

$$-(x_j) <_{lex} (g_j) <_{lex} (x_j). \tag{3.4}$$

By definition, (3.4) is equivalent to

$$\underbrace{-x_1 < g_1}_{A} \quad \text{or} \quad \underbrace{\exists n_0 \in J \ni -x_j = g_j \ \forall j < n_0 \ \text{ and } \ -x_{n_0} < g_{n_0}}_{B}$$

and

$$\underbrace{g_1 < x_1}_{C} \quad \text{or} \quad \underbrace{\exists m_0 \in J \ni g_j = x_j \ \forall j < m_0 \text{ and } g_{m_0} < x_{m_0}}_{D}$$

which is in label

$$(A \vee B) \wedge (C \vee D) = (A \wedge C) \vee (A \wedge D) \vee (B \wedge C) \vee (B \wedge D).$$

A messy computation shows that only $(B \wedge D)$ need to be considered, as the other three are impossible to occur. Furthermore, there is only a single possible subcase that needs to be considered; i.e, when $n_0, m_0 \geq i$. This implies $g_j = x_j = 0 \ \forall j < \min\{n_0, m_0\}$. But $\min\{m_0, n_0\} \geq i$. Consequently, $(g_j) \in I_i$. □

If H is a subgroup of $G = \bigoplus_{cog_{j\in J}} G_j$ with $G_j = \mathbb{Z}$ for $i \in J$, Theorem II.1.6 of [3] implies that $H = \bigoplus_{cog_{j\in J}} g_j G_j$ with $g_j \in \mathbb{Z}$.

Lemma 3.2. *If $H = \bigoplus_{cog_{j\in J}} g_j G_j$ is a non trivial c-convex subgroup of $G = \bigoplus_{cog_{j\in J}} G_j$ with $G_j = \mathbb{Z}$ for $i \in J$ such that $H \subseteq I_k$ for some $k \in J$, then there is $k_0 \in J$ with $k_0 \geq k$ such that $g_j = 0$ for all $j < k_0$, and $g_j = 1$ for all $j \geq k_0$.*

Proof. From the definition of I_k, it is clear that $g_j = 0$ for all $j < k$. Suppose $k_0 \geq k$ in J such that g_{k_0} is the first non zero number. We claim that $g_j = 1$ for all $j \geq k_0$. Suppose on the contrary that there is $j_0 \in J$ with $j_0 \geq k_0$ such that $g_{j_0} \neq 1$. Suppose $h \in H$ such that

$$h(j) = \begin{cases} g_{j_0} \text{ if } j = j_0 \\ 0 \text{ otherwise,} \end{cases}$$

and $g \in G$ such that

$$g(j) = \begin{cases} 1 \text{ if } j = j_0 \\ 0 \text{ otherwise.} \end{cases}$$

Then we get $R(-h, 0, h)$ and $R(-h, g, h)$. But $g \notin H$. This contradicts the hypothesis that H is a c-convex subgroup. Hence our assumption is not correct, and the proof is complete. □

Theorem 3.3. *Suppose $J = \{1, ..., n\}$, $G = \bigoplus_{cog_{j\in J}} G_j$ with $G_j = \mathbb{Z}, \ \forall j \in J$. If $H = \bigoplus_{cog_{j\in J}} g_j G_j$ is a non trivial c-convex subgroup of G, then $H = I_i$ for some $i \in J$.*

Proof. We claim that there is $k \in J$ such that $H \subseteq I_k$. Lemma 3.2 implies that there is $k_0 \geq k$ such that $H = I_{k_0}$ and the proof is complete. Suppose on the contrary that there is no $k \in J$ such that $H \subseteq I_k$. Since $\Sigma(G)$ is totally ordered by inclusion, $I_k \subset H$ for all $k \in J$. This implies that $g_1 \neq 0$ and $g_j = 1$ for all $j > 1$. Suppose $h \in H$ such that

$$h(j) = \begin{cases} g_1 \text{ if } j = 0 \\ 0 \text{ otherwise,} \end{cases}$$

and $g \in G$ such that

$$g(j) = \begin{cases} 1 \text{ if } j = 0 \\ 0 \text{ otherwise.} \end{cases}$$

Then $R(-h, 0, h)$ and $R(-h, g, h)$. But $g \notin H$. This contradicts the hypothesis that H is a c-convex subgroup of G. □

3.2 Finite Dimensional Free Abelian Groups

Suppose G is a finite dimensional free abelian group. A well known theorem in group theory (for example Theorem 1.1 of [3]) says that G is isomorphic to a finite direct sum of copies of additive group $\mathbb{Z}$ of integers. Recall that if $X = \{x_1, ..., x_n\}$ is a basis for G, then every element of G can be written as $\sum_{i=1}^{n} k_i x_i$ with $k_i \in \mathbb{Z}$. The following mapping is then an isomorphism between G and $\oplus_{i=1}^{n} G_i$ with $G_i = \mathbb{Z}$:

$$\psi : \sum_{i=1}^{n} k_i x_i \mapsto (k_i x_i)_{i=1}^{n}. \tag{3.5}$$

Recall that the cyclic order R for $\oplus_{i=1}^{n} G_i$ is given by (3.1). We induce a cyclic order R' for G by

$$R'(g, h, j) \iff R(\psi(g), \psi(h), \psi(j)), \tag{3.6}$$

and then G is cyclically ordered. Therefore, we get the following theorem.

Theorem 3.4. *If G is an n-dimensional free abelian group, then G is cyclically ordered under the cyclic order given by (3.6); i.e., (3.5) gives an isomorphism between G and $\oplus_{i=1}^{n}\mathbb{Z}$ as cyclically ordered groups.*

If G is a free abelian group of finite dimension, then Theorem 3.4 implies that G is cyclically ordered. Examples of c-convex subgroups of G are given by Corollary 3.5, and finally a full description of $\Sigma(G)$ is given by Corollary 3.6.

Corollary 3.5. *Suppose G is a finite dimensional free abelian group with basis $X = \{x_1, ..., x_n\}$ and $i \in \{1, ..., n\}$. Then $J_i := \{\sum_{j\geq i}^{n} \lambda_j x_j | \lambda_j \in \mathbb{Z}, x_j \in X\}$ is a c-convex subgroup of G.*

Proof. From Theorem 3.4, $\psi : \sum_{i=1}^{n} k_i x_i \mapsto (k_i x_i)_{i=1}^{n}$ gives an isomorphism between G and $\oplus_{i=1}^{n}\mathbb{Z}$. Hence each J_i in G is nothing but I_i in $\oplus_{i=1}^{n}\mathbb{Z}$. Therefore, J_i is a c-convex subgroup of G by Theorem 3.1. □

Corollary 3.6. *Suppose G is a finite dimensional free abelian group with basis $X = \{x_1, ..., x_n\}$. If H is a non-trivial c-convex subgroup of G, then $H = J_i := \{\sum_{j\geq i}^{n} \lambda_j x_j | \lambda_j \in \mathbb{Z}, x_j \in X\}$ for some $i \in \{1, ..., n\}$. Therefore, $\Sigma(G)$ consists of J_i for all $i \in \{1, ..., n\}$.*

Proof. From Theorem 3.4, $\psi : \sum_{i=1}^{n} k_i x_i \mapsto (k_i x_i)_{i=1}^{n}$ gives an isomorphism between G and $\oplus_{i=1}^{n}\mathbb{Z}$. If H is a c-convex subgroup of G, then $\psi(H)$ is a c-convex subgroup of $\oplus_{i=1}^{n}\mathbb{Z}$. Hence, from Theorem 3.3, $\psi(H) = I_i := \{(x_j)|x_j = 0 \text{ for } j < i\}$ for some $i \in \{1, ..., n\}$, therefore

$$H = \psi^{-1}(\{(x_j)|x_j = 0 \text{ for } j < i\}) = \{\textstyle\sum_{j\geq i}^{n} \lambda_j x_j | \lambda_j \in \mathbb{C}, x_j \in X\}.$$

□

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