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The relationship between chromatic polynomial and Tutte polynomial for rooted graphs and tree

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Abstract

The chromatic polynomial is a graph polynomial that is explored in algebraic graph theory. It was invented by George David Birkhoff to examine the four-color problem and it counts the number of graph colorings as a function of the number of colors. In this paper, we provide Chromatic polynomials for rooted graphs, trees, and rooted trees, as well as particular methods for their computation, the relationship between chromatic polynomial and Tutte polynomial for rooted graphs. We conclude by giving real-life examples and applications of polynomials.

1 Introduction

Graphs and their colorings are untouched top picks in starting classes on discrete mathematics. A graph is a pair of a set $G(n, m)$, where n is a finite non-empty collection of elements known as vertices and m is a finite set of elements known as edges. There is at least one directed circuit in the cycle which is a directed graph. If G is an associated graph without any cycles,

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then it is a tree [1]. For brevity, the chromatic polynomial is referred to as the "chromial". The chromatic number of a graph is the smallest positive integer that indicates the smallest number of colors with which a graph can be colored. Whitney and Tutte generalized it to the Tutte polynomial, it included several applications in software, designing, biology, physical science, engineering, and knot theory [2]. It is a two-variable polynomial that plays a significant function in graph theory. It is defined for every undirected graph G [3]. Proper coloring of graphs refers to the coloring of a graph so that no two neighboring vertices have a similar color [5]. In the 1940s, Tutte generalized Birkhof's polynomial by adding another variable and analyzing its combinatorial properties. Our work gives the relationship between achromatic polynomial and Tutte polynomial for rooted graphs which is a tree and the nodes on the leaves are cycles [6]. The trees we are discussing have a hierarchical structure in many applications [7] with one vertex on top called the root and the other vertices branching from it; for example, a computer file or library classification system is frequently organized in this way, with data stored at vertices [8]. One cycle and two cycles are discussed in the rooted graph of the nodes on the leaves in the tree. This paper is organized as follows. In the first section, the chromatic polynomial is defined. The second section deals with theorems that include the connections between Tutte polynomial and chromatic polynomial for rooted graphs containing one and two cycles. In the third section, we discuss a practical application of graph coloration known as the storage problem and the applications of rooted graphs. More applications can be found in [9-11].

2 Basic concepts

This section contains some fundamental concepts and theorems relevant to our work.

Definition1,[5] Proper Coloring of a Graph G is the process of coloring a graph in such a way that adjacent vertices are colored differently.

Definition 2,[5] The number of ways to assign k colors to the vertices of G so that no two neighboring vertices have a similar color is known as the chromatic polynomial $P(G, k)$.

Definition 3,[10] The chromatic number $\chi(G)$ is the least number of colors required to color G appropriately, and G is λ-chromatic if $\chi(G) = k$, $\chi(G) = min\{P(G, k) > 0\}$.

Definition 4,[1] The tree is a connected graph, with practically no cycles, which has n vertices and $n-1$ edges, as given in Figure 1.

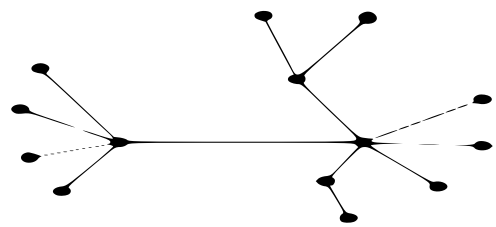

Figure 1: Tree.

Definition 5,[13] A rooted tree has a root as its specified vertex. Every edge is ostensibly pointing away from the root, as given in Figure 2.

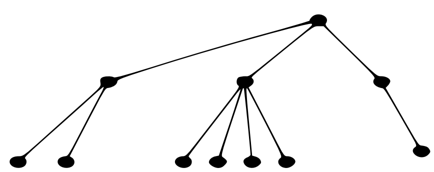

Figure 2: Rooted tree.

Definition 6,[2] A rooted graph is one in which a single point has been highlighted. The root is the name given to this distinct location, as given in Figure 3.

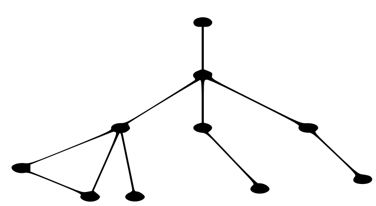

Figure 3: Rooted graph.

Definition 7,[7] A Tutte polynomial is a two-variable dichromatic polynomial which is an extension of the chromatic polynomial. Let e be an ordinary edge for the graph $G = (n, m)$. Then $T(G; x, y) = T(G - e; x, y) + T(G/e; x, y)$.
Definition 8,[14] A matroid M is a pair (X, I) in which X is a finite set and

I is a non-empty subset family of X (called the independent sets). Matroid qualities include a matroid M as a pair (X, I), where X is a finite set and I is a non-empty family of subsets of X (called the independent sets). Here are some properties of matroids:
i) The empty set is independent.
ii) All subsets of an independent set are independent.
iii) Given two independent sets A and B, and assuming that A has more elements than B, there exists an element an $a \in A \backslash B$ such that $B \cup A$ remains independent.

3 Extending of the chromatic polynomial for rooted graph

In this section, theorems that are used for a tree are proved for a rooted graph.
Theorem 1,[15] Let $G(n, m)$ be a graph with n vertices and m edges. Then the Deletion-contraction $P(G, k) = P(G - e, k) - P(G/e, k)$.

Theorem 2,[15] Let $G(n, m)$ be a graph with n vertices, m edges and c connected components $\chi(G) = k^c(-1)^{(m-c)}T(G; 1-k, 0,)$ where the chromatic polynomial $\chi(G) =$ the number of distinct proper $k-$coloring of G for all $\lambda \in N, T(G; 1-k, 0)$ the Tutte polynomial, m the edge in the rooted graph, and c the number component.

We now give the relation between chromatic polynomial and Tutte polynomial.

Theorem 3: If $G(n, m)$ is a rooted graph with $n \leq 5$ vertices and contains one cycle, then the chromatic polynomial and Tutte polynomial relationship is given by: $\chi(G) = k^c(-1)^{(m-c)}T(G; 1-k, 0)$
Proof: If $G(n, m)$ has five vertices, then there is at least one cycle and the chromatic polynomial for the graph, given in Figure 4, is $\chi(G) = k(k-1)^3(k-2)$. Three colors were needed to get the proper color. To find the Tutte polynomial, we follow the process of elimination and extraction (deletion/contraction) for finding the Tutte polynomial. This process is represented in Figure 5. Then the Tutte polynomial is $T(G) = x^4 + x^3$ Now, the relation between Tutte polynomial and chromatic polynomial is is given by:
$\chi(G) = (-1)^{(m-c)}k^cT(G; 1-k, 0)$

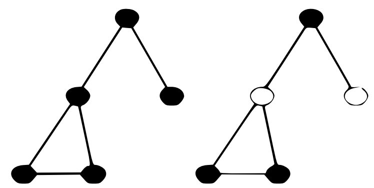

Figure 4: Rooted graph with five vertices.

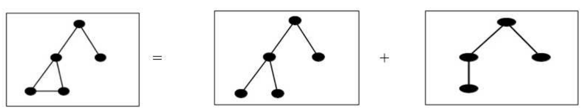

Figure 5: (Deletion/contraction) for a rooted graph with 5 vertices

$k(k-1)^3(k-2) = k(-1)^4[(1-k)^4 + (1-k)^3] = k(1-k)^3[1-k+1] = k(1-k)^3(2-k)$

Note that, the two sides are equal which means that the rooted graph has $n \leq 5$ vertices.

Theorem 4 If $G(n,m)$ is a rooted graph with $n \geq 6$ vertices and contains one cycle (one node), then the chromatic polynomial and Tutte polynomial relationship is given by:
$\chi(G) = (k/(k-1))^2(-1)^{(m-c)}k^c T(G; 1-k, 0)$

Proof: If $G(n,m)$ has $n \geq 6$ vertices and there is at least one cycle, let $f(G;k) = (k-1)^{-C}\chi(k)$.

f appears to be dependent on the graph G. However, we will prove that f is solely dependent on M_G as well as the fact that if we allow $f(M_G;k) = f(G,K)$, then this matroid function is a well-defined generalized $T-G$ invariant. $\sigma = 1$ and $\eta = -1$.If M_G isthmus (I) or loop (L), then, by linking isolated vertices, G is constructed from a single isthmus or a single loop. $f(I;k) = k-1$ and $f(L;k) = 0$. Therefore, f is well defined if $|E(T)| = 1$. Assume that e is a G edge with endpoints u and v. Assume that e isn't a loop or an isthmus in G. The set of valid $k-$colorings of $G-e$ can then be divided between those in which u and v are colored similarly and those in which they are colored differently. However, the first subset corresponds to the set of proper $k-$colorings of G/e, while the second subset corresponds to the set of proper $k-$colorings. The number of components in G and G/e is the same. G and $G-e$ have the same number of components because e is not an isthmus. Thus $f(G-e;k) = f(G;k) + f(G/e;k)$ and so, by the induction assumption, $f(G;k) = f(M_{(G-e)};k) - f(M_{(G/e)};k)$. If e is a loop of G, then $f(G;k) = f(L;k) = 0$ and so $f(G;k) = f(L;k)f(M_{(G-e)};k)$. Finally, assuming that e is an isthmus of G, the number of approaches to appropriately $k-$color G rises to the number of approaches to appropriately $k-$color $G-e$, resulting in distinct colors for u and v. However, once a color is assigned to u, there are k alternative colors that are assignable to v in a suitable $k-$coloring of $G-e$. There are a lot of ways to appropriately $k-$color $G-e$. So u and v are colored differently. $(k/(k-1))^2\chi_{(G-e)}(k)$. $\chi_G(k) = (k/(k-1))^2\chi_{(G-e)}(k)$. Since $f(I;k) = k-1$ and $C(G-e) = C(G)+1$, we conclude that $f(G;k) = f(I;k)(Ge;k)$. By the induction hypothesis, $f(G;k) = f(I;k)(M_{(G-e)};k)$. Consequently, $f(G;k)$ is well defined as a matroid function by comparing the equations for $f(G;k)$ when e is a loop, a bridge, and neither a loop nor a bridge. Since $f(I;k) = k-1andf(L,k) = 0$, $f(M_G;k) = (-1)^{(|n-c|)}T(M_G;1k,0)$. Thus $\chi_G(k) = (k/(k-1))^2(-1)^{(|n|-c)}T(M_G;1-k,0)$. Since $p(M;k) = (-1)^{(r(M))}T(M;1-k,0)$ for all matroids M.

Theorem 5 If $G(n,m)$ is a rooted graph with $n \geq 6$ vertices and contains two-cycle (two nodes in leaves for tree), then the relation between chromatic polynomial and Tutte polynomial is given by:
$\chi(G) = ((k-2)/(k-1))^2k^c(-1)^{(m-c)}T(G;1-k,0)$.

Proof: Similarly, we will give the proof of this theorem with some assumptions. There are a lot of various ways to appropriately $k-$color $G-e$ so that u and v are colored differently.$((k-2)/(k-1))^2{}_(G-e)(k)$,$\chi_G(k) = ((k-2)/(k-1))^2\chi_{(G-e)}(k)$. Since $f(I;k) = k-1$ and $C(G-e) = C(G)+1$, we conclude that $f(G;k) = f(I;k)(Ge;k)$, and so, by the induction assumption, $f(G;k) = f(I;k)(M_{(G-e)};k)$. Consequently, $f(G;k)$ is well defined as a matroid function by comparing the equations for $f(G;k)$ when e is a loop, a bridge, or neither of these. Since $f(I;k) = k-1$ and $f(L,k) = 0, f(M_G;k) = (-1)^(|n-c|)T(M_G;1k,0)$. Thus $\chi_G(k) = ((k-2)/(k-1))^2(-1)^{(|n|-c))}T(M_G;1-k,0)$. Since $p(M;k) = (-1)^{(r(M))}T(M;1-k,0)$ for all matroids M.

4 Applications

There are a lot of real-life applications of chromatic polynomial and rooted graphs in different areas of mathematics and other science [16]:
a) Scheduling Problems in Management Science.
b) Allocating Transmission Frequencies to TV and Radio Stations.
c) Coloring Maps so that no two regions that share a boundary are the same color.

d) Traffic light plan.
e) Scheduling of exams.
f) Lecture timetabling.
g) Storage problem.
(For further applications, we refer the reader to [17-19].)

4.1 Storage problem application. We start with a practical application of graph coloring known as the storage problem. Assume a college's Department of Chemistry needs to store its synthetic compounds. It is very plausible that a few synthetic substances cause rough responses when united. Such synthetic compounds are contradictory synthetic substances. For safety, incongruent synthetics should be kept in particular rooms. The simplest method for achieving this is to store one compound in each room. However, this is not the most efficient way as this requires a lot of rooms. Therefore, a natural question to ask is: What is the bare minimum of rooms required for all the synthetic compounds that in each room only compatible synthetic compounds are stored? [20].

Example (1): In the Department of Chemistry there are six different types of chemicals, say: $N, M, K, L, H,$ and T. Due to the connections among the reactants and the non-reactants, where a few synthetics can't be kept in a similar room.

Table 1: The compound table shows which synthesis cannot be together .

Type	N	M	K	L	H	T
Can't be with	M,K	N,K,H	N,M,L,H	K,T	M,K,T	L,H

What is the bare minimum of rooms required to store all chemicals? To help answer the question, we draw six points, one representing each type of chemical, and then draw a graph where each edge meets the vertices representing any two incompatible substances. Next, we determine the color number of our graph and, finally, we get the answer to the original question. We see that $\chi(G) = 3$. This means that we need three rooms: room (1) for N, H, room (2) for M, L, room(3) for K, T as in Figure 6.R

Example(2): rooted graphs. As an application, a flow diagram is an almost tree. A flowchart is a diagram that depicts an algorithm, a work flow, or a process by depicting steps as various types of boxes and linking them with arrows where we represent squares with (vertices) in the graph and arrows with (edges). We then calculate the chromatic number of the flowchart of Figure 6. Now, the chromatic number is $\chi(G) = 3$, This means that we need

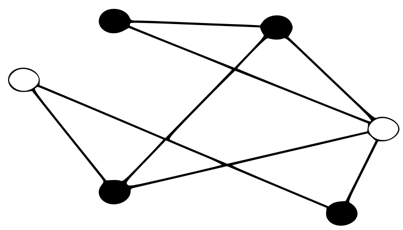

Figure 6: Storage problem.

only three instructions for the information to flow correctly for a particular class of mathematical problems as shown in Figure 7.

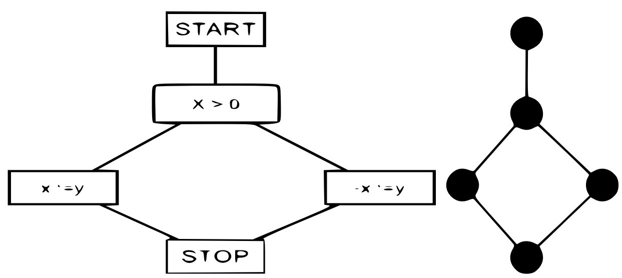

Figure 7: Convert a flowchart into a rooted graph.

5 Conclusion

Graph Theory is a discipline of mathematics with numerous outstanding issues and applications in a variety of sectors of mathematics and science. The chromatic polynomial is a type of polynomial that has useful and attractive qualities. In this paper, among other things, we proved theorems related to the rooted graph.

References

[1] R. J. Trudeau, Introduction to graph theory, Courier Corporation, 2013.

[2] G. Gary, Chromatic and Tutte polynomials for graphs, rooted graphs, and trees, Graph Theory Notes, **54,** (2008), 34–45, doi:10.1007/978-3-319-97686-0.

[3] K. Shin Min et al., Tutte polynomials with applications, Global Journal of Pure and Applied Mathematics, **12,** no. 6, (2016), 4781–4797, doi:10.48550 arXiv.2105.06798.

[4] S. E. Adams, Chromatic Polynomials for Graphs with Split Vertices, Bowling Green State University,(2020).

[5] K. Čulík, Extensions of rooted trees and their applications, Discrete Mathematics, **18,** no. 2, (1977), 131–148, doi.org/10.1016/0012-365X(77)90074-7.

[6] E. Monaghan, A. Joanna, C. Merino, Graph polynomials and their applications I: The Tutte polynomial, Structural analysis of complex networks, (2011), 219–255, doi:10.48550/arXiv.0803.3079

[7] Shatha Asaad Salman, Anwar Khaleel Faraj, Alaa Sabeeh Dawood, The Relationship between the Stasheff Polytope and Painted Trees Using Tubings, Engineering and Technology Journal, 34, no. 1, Part (B), (2016).

[8] Shatha A. Salman, Computing The Number of Integral Points in 4-dimensional Ball Using Tutte Polynomial, Engineering and Technology Journal, **33,** (2015), 1420–1429.

[9] Shatha A. Salman, On the number of lattice points in n-dimensional space with an application, Computer Science, **16,** no. 2, (2021), 705–712.

[10] R. J. Wilson, Introduction to graph theory, 1979.

[11]] Shatha A. Salman, Abeer H. Abd-Almeer, Clotting detection in the vascular network, Indonesian Journal of Electrical Engineering and Computer Science, **20,** no. 2, (2020), 1103–1108, doi:10.11591/ijeecs.v20.i2.

[12] Shatha A. Salman, Abeer H. Abd-Almeer, The Minimum Cost Connected Subgraph for the Vascular Network, Energy Procedia, **157,** (2019), 128–134, doi: 10.1016/j.egypro.2018.11.172.

[13] Béal Sylvain, Eric Rémila, Philippe Solal, Rooted-tree solutions for tree games," European Journal of Operational Research, **203,** no. 2, (2010), 404–408.

[14] D. L. Neel, N. Ann Neudauer, Matroids you have known, Mathematics magazine, **82,** no. 1, (2009), 26–41.

[15] T. Brylawski, J. Oxley, The Tutte polynomial and its applications, Matroid applications, **40,** (1992), 123–225, doi:10.1017/CBO9780511662041.007.

[16] D. Narmatha, D. Indhumathy, Applications of Graph Coloring-Palarch, Journal Of Archaeology Of Egypt/Egyptology, **17,** no. 7.

[17] Shatha A. Salman, Abeer Hussin, The Minimum Cost for the Vascular Network using linear programming based its path graph, Iraqi journal of science, **60,** no. 4, (2019), 859–867.

[18] Shatha A. Salman, A. S. Fatiem, The Ehrhart polynomials of the cyclic polytope, Engineering and Technology Journal, **14,** no. 27, (2009), 2624.

[19] Nuha A. Rajab, Samaa F. Ibraheem, Eman H. Ouda, On Some Properties of Characteristics Polynomials of the Complete Graphs Kn, Engineering and Technology Journal, (2013), **31**, no. 4, Part (B), 520–528.

[20] R. Balakrishnan, K. Ranganathan, A textbook of graph theory, Springer Science and Business Media, 2012. doi:10.1007/978-1-4614-4529-6.