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Solutions of the Diophantine Equation $p^x + q^y = z^2$, where $p, q \equiv 3 \pmod 4$

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Abstract

In this article, we study the Diophantine equation $p^x+q^y = z^2$ with $p, q > 3$ and $p, q \equiv 3 \pmod 4$ and x, y, z are non-negative integers. We give some conditions for the non-existing of non-negative integer solutions and show all solutions for $x = 1$. Moreover, we apply our results to investigate the solutions of the Diophantine equation $p^x + 11^y = z^2$.

1 Introduction

Finding non-negative integer solutions of the Diophantine equation $p^x+q^y = z^2$, where p and q are prime numbers, is a famous topic in number theory. Many researchers have solved the equation when p and q are explicit prime numbers (see, for example, [4], [5] and [11]). Moreover, Cheenchan et al. [6] have investigated the Diophantine equation by giving p or q as variables with some conditions such as $p^x + 5^y = z^2$.

Burshtein [2] studied the Diophantine equation $2^x + p^y = z^2$. In 2021, Tangjai and Chubthaisong [12] gave non-negative integer solutions for the Diophantine equation $3^x + p^y = z^2$, where $p \equiv 2 \pmod 3$. One year later,

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Pakapongpun and Chattae [9] gave the solutions of the Diophantine equation $p^x + 7^y = z^2$ with $p \equiv 1 \pmod 6$ or $p \equiv 3 \pmod 6$ or $p \equiv 5 \pmod 6$ with some conditions. Our first aim was to study the Diophantine equation $p^x + 11^y = z^2$. However, it turned out that we could find the solution to a generalized equation. Consequently, the purpose of this work is to study the solutions of the Diophantine equation $p^x + q^y = z^2$ with $p, q > 3$ and $p, q \equiv 3 \pmod 4$.

2 Main results

In this work, we consider the Diophantine equation $p^x+q^y = z^2$ with $p, q > 3$, $p, q \equiv 3 \pmod 4$, and $p \neq q$ (Burshtein [3] have studied the solutions when $p = q$). If $x = 0$, then $1 + q^y = z^2$ or $z^2 - q^y = 1$ from which it follows that $q = 2$ using Catalan's conjecture, proven by Mihailescu[7]: (3,2,2,3) is the unique solution (a, b, x, y) for the Diophantine equation $a^x - b^y = 1$ where a, b, x and y are integers such that $\min\{a, b, x, y\} > 1$ but this contradicts $q > 3$. Similarly, if $y = 0$, then $p = 2$, a contradiction. As a result, for $x = 0$ or $y = 0$ the equation $p^x + q^y = z^2$ has no non-negative integer solution. Therefore, from now on, p and q are distinct prime numbers with $p, q > 3$, $p, q \equiv 3 \pmod 4$ and x, y are positive integers.

The following results explain the relation of x and y when the Diophantine equation has a positive integer solution.

Theorem 2.1. *If the Diophantine equation $p^x+q^y = z^2$ has a positive integer solution, then x and y have opposite parity.*

Proof. Assume that the Diophantine equation $p^x + q^y = z^2$ has a positive integer solution. Since $p \equiv 3 \pmod 4$ and $q \equiv 3 \pmod 4$, we have $p^x + q^y \equiv (-1)^x + (-1)^y \pmod 4$. So $z^2 \equiv (-1)^x + (-1)^y \pmod 4$. Since p and q are odd numbers, we have $p^x + q^y$ is an even number. Then $z^2 \equiv 0 \pmod 4$ and $(-1)^x + (-1)^y \equiv 0 \pmod 4$. Therefore, x and y have opposite parity.

Theorem 2.2. *If the Diophantine equation* $p^x+q^y = z^2$ *has a positive integer solution and* y *is an even number, then* $2q^{\frac{y}{2}} = p^x - 1$ *and* $p \equiv 1 \pmod q$*.*

Proof. Assume that the Diophantine equation $p^x + q^y = z^2$ has a positive integer solution and y is an even number. Then there exists a positive integer k such that $y = 2k$. Thus $p^x + q^{2k} = z^2$ so that $p^x = z^2 - q^{2k} = (z - q^k)(z + q^k)$. Since p is a prime number, there exists a non-negative integer u with $u \leq x$ such that $z - q^k = p^u$ and $z + q^k = p^{x-u}$. Then $x > 2u$ and $2q^k = p^u(p^{x-2u} - 1)$. Since p and q are distinct and $p \neq 2$, we have $u = 0$. Thus $2q^k = p^x - 1 = (p - 1)(p^{x-1} + p^{x-2} + \cdots + 1)$. Therefore, $q|(p - 1)$.

Similarly, we obtain the following result.

Theorem 2.3. *If the Diophantine equation* $p^x+q^y = z^2$ *has a positive integer solution and* x *is an even number, then* $2p^{\frac{x}{2}} = q^y - 1$ *and* $q \equiv 1 \pmod p$*.*

Next, we consider the Diophantine equation $p^x + q^y = z^2$, where $p \not\equiv 1 \pmod q$ and $q \not\equiv 1 \pmod p$.

Theorem 2.4. *The Diophantine equation* $p^x+q^y = z^2$*, where* $p \not\equiv 1 \pmod q$ *and* $q \not\equiv 1 \pmod p$*, has no positive integer solution.*

Proof. By Theorem 2.2 and $p \not\equiv 1 \pmod q$, we have y is an odd number or the Diophantine equation has no positive integer solution. Suppose that the Diophantine equation has positive integer solution. Then y is an odd number. So x is an even number by Theorems 2.1 and 2.3. Therefore, $q \equiv 1 \pmod p$, a contradiction.

The previous theorem shows that many Diophantine equations such as $7^x + 11^y = z^2$ [5], $7^x + 31^y = z^2$ [10], $7^x + 19^y = z^2$ and $7^x + 91^y = z^2$ [11], $p^x + (p + 4)^y = z^2$ where $p \equiv 3 \pmod 4$ [1] and $p^x + (p + 8)^y = z^2$ where $p \equiv 3 \pmod 4$ [8] have no positive integer solution.

Next, we study the conditions $p \equiv 1 \pmod q$ or $q \equiv 1 \pmod p$. In fact, these two conditions do not exist in the same situation. The condition $p \equiv 1 \pmod q$ will exist when $p > q$.

Theorem 2.5. *The Diophantine equation* $p^x+q^y=z^2$, *where* $p \equiv 1 \pmod{q}$ *and* x *is an even number, has no positive integer solution.*

Proof. We have $2p^{\frac{x}{2}} = q^y - 1$ by Theorem 2.3. Since $p \equiv 1 \pmod{q}$, we have $2p^{\frac{x}{2}} \equiv 2 \pmod{q}$ and $q^y - 1 \equiv 2 \pmod{q}$. Then $q = 3$, a contradiction.

By the above result, we obtain Theorem 2.1 in [1], the Diophantine equation $11^x+23^y=z^2$ has no positive integer solution when y is an even number. Next, we find the solutions of the Diophantine equation where x is an odd number. First, let $x = 1$.

Theorem 2.6. *If the Diophantine equation* $p+q^y=z^2$ *has a positive integer solution, then* y *is an even number and* $(p, q, y, z) = (2q^{\frac{y}{2}} + 1, q, y, q^{\frac{y}{2}} + 1)$.

Proof. By Theorems 2.1 and 2.2, we obtain y is an even number and $2q^{\frac{y}{2}} = p - 1$, respectively. Thus $z^2 = p + q^y = 2q^{\frac{y}{2}} + 1 + q^y = (q^{\frac{y}{2}} + 1)^2$. Then $z = q^{\frac{y}{2}} + 1$. Therefore, this equation has solutions $(p, q, y, z) = (2q^{\frac{y}{2}} + 1, q, y, q^{\frac{y}{2}} + 1)$.

In the following theorem, we find a relation between q and x when the Diophantine equation $p^x + q^y = z^2$ has a positive integer solution and x is an odd number with $x > 1$.

Theorem 2.7. *Let* x *be an odd number and* $x > 1$. *If the Diophantine equation* $p^x + q^y = z^2$ *has a positive integer solution, then* $q|x$.

Proof. Since x is an odd number, we have y is an even number by Theorem 2.1. By Theorem 2.2, we obtain $2q^{\frac{y}{2}} = p^x - 1$ and $p \equiv 1 \pmod{q}$. Then $2q^{\frac{y}{2}} = (p-1)(p^{x-1} + p^{x-2} + \cdots + 1)$. Since $x > 1$, $y > 0$ and $p \equiv 1 \pmod{q}$, we have $q|(p^{x-1} + p^{x-2} + \cdots + 1)$ and $p^{x-1}, p^{x-2}, p^{x-3}, \ldots, p \equiv 1 \pmod{q}$. So $p^{x-1} + p^{x-2} + \cdots + 1 \equiv x \pmod{q}$. Thus $q|x$.

By Theorems 2.1 and 2.7, we obtain the following theorem.

Theorem 2.8. *Let* $x, y > 1$. *If the Diophantine equation* $p^x + q^y = z^2$ *has a positive integer solution, then* $q|x$ *or* $p|y$.

From the last theorem, we have a condition for the non-existence of positive integer solutions of the Diophantine equation $p^x + q^y = z^2$.

We conclude our paper with the following result about the Diophantine equation $p^x + 11^y = z^2$, where $p \equiv 3 \pmod 4$ and $p > 3$.

- The Diophantine equation $p^x + 11^y = z^2$ has no non-negative integer solution, where $x = 0$ or $y = 0$ or $p \not\equiv 1 \pmod{11}$ or $p \equiv 1 \pmod{11}$ and x is an even number.

- The Diophantine equation $p + 11^y = z^2$ has positive integer solutions $(p, y, z) = (2(11^{\frac{y}{2}}) + 1, y, 11^{\frac{y}{2}} + 1)$; for example, $23 + 11^2 = 12^2$ and $2663 + 11^6 = 1332^2$.

- If the Diophantine equation $p^x + 11^y = z^2$, where x is an odd number and $x > 1$, has a positive integer solution, then $11|x$.

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