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On the Diophantine Equation $3^x + b^y = z^2$

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Abstract

In this paper, we find all non-negative integer solutions (x, y, z) of the Diophantine equation $3^x + b^y = z^2$, where b is a positive integer such that $b \equiv 5(\text{mod}20)$ or $b \equiv 5(\text{mod}30)$. If $b \equiv 5(\text{mod}20)$, then the solution (x, y, z) of the equation is $(1, 0, 2)$. In case that $b \equiv 5(\text{mod}30)$, we get $(x, y, z) \in \{(1, 0, 2), (0, 1, \sqrt{b+1}) \mid \sqrt{b+1} \in \mathbb{Z}\}$.

1 Introduction

A Diophantine equation is an equation in which only an integer solution is allowed, when the equation is solvable. The well-known Diophantine equation of the form $a^x + b^y = z^2$, where a and b are positive integers has been studied by many researchers.

In 2012, Sroysang [1] proved that $(1, 0, 2)$ is the unique solution (x, y, z) for the Diophantine equation $3^x + 5^y = z^2$, where x, y and z are non-negative

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integers.

Moreover, in 2013, Sroysang [2, 3] also proved that $\{(0,1,2),(3,0,3),(4,2,5)\}$ and $\{(1,0,2)\}$ are the set of all non-negative integer solutions (x,y,z) for the Diophantine equation $2^x + 3^y = z^2$ and $3^x + 17^y = z^2$, respectively. Furthermore, Rabago [4] studied the solutions (x,y,z) of the two Diophantine equations $3^x+19^y = z^2$ and $3^x+91^y = z^2$ and found that $\{(1,0,2),(4,1,10)\}$ and $\{(1,0,2),(2,1,10)\}$ are the set of all non-negative integer solutions for the above equations, respectively.

In [5, 6, 7, 8, 9], one can see non-negative integer solutions of the Diophantine equation $3^x + p^y = z^2$, when p is a prime number.

Moreover, finding a non-negative integer solution of the Diophantine equation $3^x + b^y = z^2$, for some positive integer b is also interesting. For example, in 2014, Sroysang [10, 11] solved the equations $3^x + 85^y = z^2$ and $3^x + 45^y = z^2$ (85 and 45 are not prime numbers) and found that $(1,0,2)$ is the unique solution in non-negative integer x, y and z for these two equations.

Inspired by the work mentioned earlier, we study the Diophantine equation $3^x + b^y = z^2$, where b is a positive integer such that $b \equiv 5(\mathrm{mod}20)$ or $b \equiv 5(\mathrm{mod}30)$ and b doesn't have to be a prime number.

2 Preliminaries

First, we give some helpful Theorem and Lemmas for this study as follows.

Theorem 2.1 ([12](Mihailescu's Theorem)). *$(3,2,2,3)$ is the unique solution (a,b,x,y) for the Diophantine equation $a^x - b^y = 1$, where a,b,x and y are integers such that $min\{a,b,x,y\} > 1$.*

Lemma 2.2 ([1, Lemma2.2]). *$(1,2)$ is the unique solution (x,z) for the Diophantine equation $3^x + 1 = z^2$, where x and z are non-negative integers*

Lemma 2.3. *Let b be a positive integer such that $b \equiv 5(\mathrm{mod}20)$. Then the Diophantine equation $1 + b^y = z^2$ has no non-integer solution.*

Proof. Suppose that there are non-negative integers y and z such that $1+b^y = z^2$. If $y = 0$, then $z^2 = 2$, which is a contradiction. Thus $y \geq 1$.

Case 1: $y = 1$. We get $1 + b = z^2$. Since $b \equiv 5 (\text{mod} 20)$, $b \equiv 1 (\text{mod} 4)$. We have $z^2 \equiv 1 + b \equiv 2 (\text{mod} 4)$, which contradicts the fact that $z^2 \equiv 0, 1 (\text{mod} 4)$.

Case 2: $y > 1$. It is easy to check that $b > 1$ and $z > 1$. Since $z^2 - b^y = 1$, we get $b = 2$ by Theorem 2.1. Hence $2 \equiv 5 (\text{mod} 20)$. This is the desired contradiction. □

Lemma 2.4. *Let b be a positive integer such that $b \equiv 5 (\text{mod} 30)$. Then $(y, z) = (1, \sqrt{b+1})$ represent the non-negative integer solutions of the Diophantine equation $1 + b^y = z^2$, where $\sqrt{b+1}$ is an integer.*

Proof. Let y and z be non-negative integers such that $1 + b^y = z^2$. If $y = 0$, then $z^2 = 2$, which is a contradictions. Thus $y \geq 1$.

Case 1: $y = 1$. We get $(y, z) = (1, \sqrt{b+1})$ is a solution of the Diophantine equation $1 + b^y = z^2$, where $\sqrt{b+1}$ is an integer.

Case 2: $y > 1$. It is easy to check that $b > 1$ and $z > 1$. Since $z^2 - b^y = 1$, we get $b = 2$ by Theorem 2.1. Thus $2 \equiv 5 (\text{mod} 30)$, a contradiction. This finishes the proof.

□

3 Main results

In this section, we consider solutions of the Diophantine equation $3^x + b^y = z^2$, where x, y and z are non-negative integers.

Theorem 3.1. *Let b be a positive integer such that $b \equiv 5 (\text{mod} 20)$. Then $(1, 0, 2)$ is the unique solution (x, y, z) for the Diophantine equation $3^x + b^y = z^2$, where x, y and z are non-negative integers.*

Proof. Let x, y and z be non-negative integers such that $3^x + b^y = z^2$. Since b is odd, $3^x + b^y$ is even and so z^2 is even. Thus z is even. Hence $z^2 \equiv 0 (\text{mod} 4)$. This implies that $3^x + b^y \equiv 0 (\text{mod} 4)$. Since $b \equiv 5 (\text{mod} 20)$, $b \equiv 1 (\text{mod} 4)$. We get $b^y \equiv 1 (\text{mod} 4)$. This implies that $3^x \equiv 3 (\text{mod} 4)$. Thus $(-1)^x \equiv -1 (\text{mod} 4)$. So x is odd.

Consider the following cases:

Case 1: $y = 0$. By Lemma 2.2, the solution (x, y, z) of the Diophantine equation is $(1, 0, 2)$.

Case 2: $y \geq 1$. Since $b \equiv 5(\text{mod}20)$, $b \equiv 0(\text{mod}5)$ and so $b^y \equiv 0(\text{mod}5)$. Since x is odd, it follows that $3^x \equiv 2, 3(\text{mod}5)$. Thus $z^2 \equiv 3^x + b^y \equiv 2, 3(\text{mod}5)$ which contradicts the fact that $z^2 \equiv 0, 1, 4(\text{mod}5)$. This completes the proof.

□

It can be seen that our result generalizes the result of Sroysang [1, 10, 11], as the following corollaries show.

Corollary 3.2. *The Diophantine equation* $3^x + 5^y = z^2$ *has the unique non-negative integer solution* $(x, y, z) = (1, 0, 2)$.

Corollary 3.3. *The Diophantine equation* $3^x + 45^y = z^2$ *has the unique non-negative integer solution* $(x, y, z) = (1, 0, 2)$.

Corollary 3.4. *The Diophantine equation* $3^x + 85^y = z^2$ *has the unique non-negative integer solution* $(x, y, z) = (1, 0, 2)$.

Next, we give a non-negative integer solution of the Diophantine equation $3^x + b^y = z^2$, when $b \equiv 5(\text{mod}30)$:

Theorem 3.5. *Let* b *be a positive integer such that* $b \equiv 5(\text{mod}30)$. *Then all non-negative integer solutions of the Diophantine equation* $3^x + b^y = z^2$ *are* $(x, y, z) \in \{(1, 0, 2), (0, 1, \sqrt{b+1}) \mid \sqrt{b+1} \in \mathbb{Z}\}$.

Proof. Let x, y and z be non-negative integers such that $3^x + b^y = z^2$.

Consider the following cases:

Case 1: $x = 0$. By Lemma 2.4, we get $(x, y, z) = (0, 1, \sqrt{b+1})$ is a solution of the Diophantine equation $3^x + b^y = z^2$, where $\sqrt{b+1}$ is an integer.

Case 2: $x \geq 1$. Since $3^x \equiv 0(\text{mod}3)$ and $b \equiv -1(\text{mod}3)$, $z^2 \equiv 3^x + b^y \equiv (-1)^y(\text{mod}3)$. Since $z^2 \equiv 0, 1(\text{mod}3)$, we have $(-1)^y \equiv 1(\text{mod}3)$ and so y is an even. Let $y = 2k$, for some non-negative integer k. We get $3^x = (z - b^k)(z + b^k)$. Then there exists a non-negative integer u such that $z - b^k = 3^u$ and $z + b^k = 3^{x-u}$, where $x \geq 2u$. Thus $2b^k = 3^u(3^{x-2u} - 1)$. Since $b \equiv 2(\text{mod}3)$, $u = 0$ and so $2b^k = 3^x - 1$. If $k = 0$, then $(x, y, z) = (1, 0, 2)$. In case of $k \geq 1$, we have $b \equiv 0(\text{mod}5)$ and so $2b^k \equiv 0(\text{mod}10)$. Hence $3^x \equiv 2b^k + 1 \equiv 1(\text{mod}10)$. Since $3^x + b^y = z^2$ and $b \equiv 5(\text{mod}30)$, z is even. It follows that $z^2 \equiv 0(\text{mod}4)$. We have $b \equiv 1(\text{mod}2)$. Then $b \equiv \pm1(\text{mod}4)$. Since y is an even, the only possible case is $b^y \equiv 1(\text{mod}4)$.

Thus $(-1)^x \equiv 3^x \equiv z^2 - b^y \equiv -1 (\mathrm{mod} 4)$. It follows that x is an odd. Hence $3^x \equiv 3, 7 (\mathrm{mod} 10)$. This contradicts the fact that $3^x \equiv 1 (\mathrm{mod} 10)$. This completes the proof. □

Corollary 3.6. *The Diophantine equation* $3^x + 35^y = z^2$ *has only two non-negative integer solutions* $(x, y, z) \in \{(1, 0, 2), (0, 1, 6)\}$.

Corollary 3.7. *The Diophantine equation* $3^x + 65^y = z^2$ *has the unique non-negative integer solution* $(x, y, z) = (1, 0, 2)$.

Corollary 3.8. *The Diophantine equation* $3^x + 575^y = z^2$ *has only two non-negative integer solutions* $(x, y, z) \in \{(1, 0, 2), (0, 1, 24)\}$.

4 Conclusions

In this paper, we obtained all non-negative integer solutions (x, y, z) of the Diophantine equation $3^x + b^y = z^2$, where b is a positive integer such that $b \equiv 5 (\mathrm{mod} 20)$ or $b \equiv 5 (\mathrm{mod} 30)$. As a result, we have generalized the research articles of Sroysang in [1, 10, 11]. Nevertheless, to find all the solutions of the Diophantine equation $3^x + b^y = z^2$ for any integer b is still open problem.

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