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Semiprime Hollow R-modules

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Abstract

The idea of uniform modules was utilized to create what is now known as the uniform dimension of a module (also known as the Goldie dimension). Some characteristics of the idea of the dimension of a vector space are generalized by uniform dimension. Let R be an identity commutative ring and let Z be an R-module. We examine some main properties of Semiprime Hollow R-Modules, as well as the relationship between them and hollow modules and other modules like semihollow, abundantly supplemented, and lifting modules.

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1 Introduction

Assume R is an identity commutative ring and Z is an R-module. Q is called a small submodule of Z and denoted by $Q \ll Z$, if $Q+U=Z$ for any proper submodule U of Z implies $U=Z$ [1,2]. Any proper submodule Q of Z is called a semiprime submodule, if $r^k x \in Q$ where $r \in R, k \in Z^+$ and $x \in Z$, then $rx \in Q$ [3]. A non-zero R-module Z is called hollow (H-R-M), if every proper submodule of Z is small [1]. Any submodule S of Z is called coclosed if for each submodule $L \leq Z$ with $L \subseteq S$, then $\frac{S}{L} \ll \frac{Z}{L}$, then $S=Z$ [4]. An R-module Z is called a small cover for an R-module T, if there exists a small epimorphism $\phi : T \to Z$ [5]. The R-module Z is projective, if for any epimorphism $\alpha : H_1 \to H_2$ where H_1 and H_2 are R-modules, and for any homomorphism $\beta : Z \to H_2$ there exists a homomorphism $\theta : Z \to H_1$ such that $\alpha \circ \theta = \beta$ [6]. The R-module Z is C.P module if every cyclic submodule of Z is projective [7].

2 Semiprime Hollow R-Modules

This section introduces and investigates certain features and characterizations of a newly SP-H-R-M generalization:

Definition 2.1 The R-module Z is semiprime hollow (SP-H-R-M) if for any semiprime submodule of Z is small.

Remarks and Examples 2.2

1. In general, any H-R-M is SP-H-R-M. However, the opposite is not true. A Z-module Q is known to be not H-R-M [15].

2. In general, any prime H-R-M is also a SP-H-R-M, but that isn't always the case. A Z-module Q is known to be not H-R-M [15].

3. Z-module Z is not SP-H-Z-M. Since every semiprime submodule of Z is form (p), where p a prime number, but (p) isnt a small submodule of Z.

4. The module Z_9 as Z-module is semiprime hollow module. The only semiprime submodule of Z_9 is $E=(\bar{3})$ which is small submodule of Z_9 and there is no proper submodule K of Z_9 such that $E+K=Z_9$.

Proposition 2.3 Every SP-H-R-M created finitely generated is H-R-M.
Proof: Let Z be a finitely generated R-module. Then any proper submodule S of Z is contained within a maximal submodule W. Since Z is SP-H-R-M, $W \ll Z$. Hence $S \ll Z$. Thus Z is H-R-M.

Proposition 2.4 Let Z_1, Z_2 be any R-modules also $\psi : Z_1 \to Z_2$ be an R-epimorphism. If Z_1 is a SP-H-R-M, then Z_2 is also SP-H-R-M.
Proof: It is clear that $\psi(Z_1) = Z_2$, since ψ is an R-epimorphism. Let S be a semiprime submodule of Z_2. We prove that $S \ll Z_2$. Since $\psi^{-1}(S) \leq Z_2$, $\psi{-}1(S)$ is semiprime submodule of Z_1 [15] and also Z_1 is a SP-H-R-M. Then $\psi{-}1(S) \ll Z_2$. Now, $\psi(\psi^{-1}(S)) \ll \psi(Z_1) = Z_2$, [6] and hence $S \ll Z_2$. Thus Z_2 is SP-H-R-M.
Corollary 2.5 If S is a proper submodule in SP-H-R-M Z, then $\frac{Z}{S}$ is SP-H-R-M.
Corollary 2.6 The direct summand of SP-H-R-M is SP-H-R-M.
Proof : Let Z be a SP-H-R-M and $E, F \leq Z$ such that $E \oplus F = Z$. The projections $P_E : Z \to E$ and $P_F : Z \to F$ are R-epimorphism R-modules. By Proposition(2.4), we get E and F are semiprime hollow.

Proposition 2.7 If Z is semiprime finitely generated H-R-M, then it is a cyclic R-module.
Proof : By Proposition(2.3), Z is H-R-M and hence Z is cyclic R-module.

Proposition 2.8 If S is a semiprime submodule of SP-H-R-M and Z also $\frac{Z}{S}$ is finitely generated, then Z is H-R-M.
Proof: Now to show that Z is finitely generated, as $\frac{Z}{S}$ is finitely generated, there exist $x_1, x_2, ..., x_n \in Z$ and $\frac{Z}{S} = Rx_1, Rx_2, ..., Rx_n + S$. Let $x \in Z, x + S \in \frac{Z}{S}$ and there exist $a_1, a_2, ..., a_n \in Z$ such that $x + S = a_1x_1, a_2x_2, ..., a_nx_n + S$. Then $x - a_1x_1, a_2x_2, ..., a_nx_n = s, s \in S$. Hence $Z = Rx_1, Rx_2, ..., Rx_n + S$. Since Z is SP-H-R-M and $\frac{Z}{S}$ is finitely generated also by Proposition(2.3). Thus Z is H-R-M.

Proposition 2.9 Let Z be SP-H-R-M. If a proper submodule is semiprime in Z, then any non-zero coclosed submodule of Z is SP-H-R-M.
Proof: Assume S is non-zero coclosed submodule of Z and K is a proper submodule of S. Then $K \subset S$. Since Z is SP-H-R-M and K is prime submodule of Z also $K \ll Z$. But S is coclosed submodule of Z. Thus $K \ll Z$, see [4, P.27].

Proposition 2.10 Let V be a small cover of an R-module Z. If V is SP-H-R-M, then Z is SP-H-R-M.
Proof: Let $\phi : V \to Z$ be a small cover of Z and V is a SP-H-R-M. By the first isomorphism theorem $\frac{V}{Ker(\phi)} \cong Z$ and by **Corollary(2.5)**. Thus Z is SP-H-R-M.

Corollary 2.11 If Z is SP-H-R-M and is a finitely generated C.P module, then Z is projective.
Proof: By Proposition 2.7, Z is cyclic. But Z is a C.P module and thus Z is projective module.

3 More About Semiprime Hollow R-Modules

SP-H-R-M is studied in this section. An R-module Z is indeed a multiplication R-module (M-R-M) if for each submodule Q of Z there exists an ideal J of R such that $Q = JZ$ [11].

Proposition 3.1 Let Z be M-R-M containing a finitely generated semiprime submodule of an R-module Z. If Z is SP-H-R-M, then Z is H-R-M.
Proof: Since Z is M-R-M containing a finitely generated semiprime submodule, Z is finitely generated [12], and by **Proposition 2.3**, we get the result.

Corollary 3.2 If Z is a M-R-M with a semiprime annihilator and SP-H-R-M, then Z is H-R-M.
Proof: Since Z is a M-R-M. with semiprime annihilator, Z is finitely generated [12] and hence Z is H-R-M.

Theorem 3.3 If R is semiprime hollow ring and Z is multiplication finitely generated and faithful module over R, then Z is semiprime hollow module.
Proof: Assume that Q is a semiprime submodule of Z. As Z is M-R-M, there exists a semiprime ideal J in R so that $Q = JZ$ [13]. But R is semiprime hollow ring. Thus J is small ideal in R. As Z is faithful finitely generated and M-R-M, Q is a small submodule of Z [14].
Recall that the R-module Z is cancellation R-module, if whenever $JZ = LZ$ where J and L be two ideals of R, then $J = L$. Also Z is called weak cancellation, if whenever $JZ = LZ$, where J and L be two ideals of R then

$J+ann(Z)=L+ann(Z)$. Also Z is called quasi-cancellation module, if whenever $JZ=LZ$ where J and L be two finitely generated ideals of R then $J=L$ [12]. In [12], Mijbass proved that if Z is multiplication and cancellation R-module (M-C-R-M), then Z is finitely generated and faithful.
Corollary (3.4): Let Z be a weak cancellation and multiplication R-module. If Z is SP-H-R-M, then Z is H-R-M.

Proposition 3.5 Let Z be M-C-R-M. If Z is SP-H-R-M, then Z is cyclic R-module.
Proof: Since Z is M-C-R-M, Z is finitely generated (12)and Z is SP-H-R-M. By Corollary (2.11), Z is cyclic.

Recall that with a submodule Q of an R-module Z and I an ideal in R, Q is pure in Z, if $IZ\bigcap Q=IQ$ [15].

Corollary 3.6 Let Z be an M-R-M. such that Z contains a pure weak cancellation submodule T with $ann(Z)=ann(T)$. If Z is SP-H-R-M, then Z is H-R-M.

Corollary 3.7 Let Z be an M-R-M such that Z contains a pure cancellation submodule. If Z is SP-H-R-M, then Z is H-R-M.

Proposition 3.8 Let Z be multiplication faithful over integral domain R. If Z is SP-H-R-M., then Z is H-R-M.
Proof: Since Z is multiplication faithful over integral domain R, Z is finitely generated [12], and by **Proposition 2.3**, Z is H-R-M.

Proposition 3.9 Let Z be M-R-M which has a finitely generated faithful submodule Q. If Z is SP-H-R-M, then Z is H-R-M.
Proof: Since Z is M-R-M and the submodule Q of Z is finitely generated faithful, Z is finitely generated [12]. Hence Z is H-R-M.

Proposition 3.10 Every finitely generated SP-H-R-M is lifting R-module.

Remark 3.11 In general, the opposite of Proposition(3.10) is not true; for example, the Z-module $Z=Z_2\bigoplus Z_4$ is a lifting Z-module. But it is not SP-H-Z-M, since there exists a semiprime submodule $Q=Z_2\bigoplus(0)$ of $Z_2\bigoplus Z_4$ which is not a small submodule of $Z_2\bigoplus Z_4$.

Proposition 3.12 Let Z be a faithful multiplication over an integral domain R. If Z is SP-H-R-M, then Z is a lifting R-module.
Proof: Since Z is a faithful multiplication R-module over an integral domain, Z is finitely generated [12]. Since Z is SP-H-R-M, by Proposition 3.10, Z is a lifting R-module.

Corollary 3.13 Let Z be a non-zero M-R-M with a semiprime annihilator. If Z is SP-H-R-M, then Z is lifting R-module.

Proposition 3.14 If an R-module Z is finitely generated faithful and multiplication over a semiprime hollow ring, then Z is lifting R-module.
Proof: Since Z is a finitely generated faithful and multiplication module over an integral domain, by Theorem(3.3) and Proposition 3.10, Z is lifting R-module.

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