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Friends of Two Times a Prime Number

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Abstract

We show that if $p > 3$ is a prime, and $n \neq 2p$ is a positive integer with the same abundancy index as $2p$, then n has at least three distinct prime factors, including p. Further, if n has precisely three distinct prime factors, then its smallest prime factor is 3. If n has precisely four distinct prime factors, then its smallest prime factor is 3 or 5; if the smallest is 5, then the second smallest is 7 and the third smallest is less than or equal to 31. This improves a result of Haenel and Wood [1].

1 Introduction

For a positive integer n, the abundancy index $I(n)$ is defined as $\frac{\sigma(n)}{n}$, where $\sigma(n)$ is the sum of the positive divisors of n. Two positive integers m and n are *friends* if and only if $I(m) = I(n)$ and $m \neq n$. A number that has one or more friends is called a *friendly number*, while a number that is known to have no friends is called a *solitary number*. The most classic example of a set of friendly numbers is the perfect numbers, which by definition are equal to half the sum of their positive divisors. All perfect numbers are friends with abundancy index 2.

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Lots of research has been done on friends of specific integers like 10 [3]. Additionally, Haenel and Wood [1] investigated the more general case of friends of products of two primes, as well as friends of four times a prime number. However, much is still unknown about friends of even the smallest positive integers: of the positive integers less than 100, the status (friendly or solitary) of thirty-one of them is not certain [5].

Many of these numbers with unknown status are of the form $2p$ for some prime $p > 3$ (Note that if $p = 3$, then $2p = 6$ is a perfect number and is therefore friendly. If $p = 2$, then $2p = 4$ is solitary; this will be shown later). This paper builds on [1] and [3] to establish necessary properties of friends of two times an odd prime other than 3, if such friends exist. First, some basic properties of the abundancy index function must be shown.

Properties of the Abundancy Index

These properties will be used to establish conditions for the existence of friends of $2p$. Proofs can be found in [2], [3], and [4].

1. $I(n) \geq 1$ with equality iff $n = 1$.

2. If $m|n$, then $I(m) \leq I(n)$ with equality iff $m = n$.

3. Abundancy index of products of primes: If p_1,p_2,...,p_k are the distinct prime factors of n with corresponding positive integer exponents e_1,e_2,...,e_k, then $I(\prod_{j=1}^{k} p_j^{e_j}) = \prod_{j=1}^{k} \frac{p_j^{e_j+1}-1}{p_j^{e_j}(p_j-1)}$. This is derived from Property 4 below and from the well-known formula $\sum_{j=1}^{k} r^j = \frac{r^{k+1}-1}{r-1}(r \neq 1)$.

4. The abundancy index is weakly multiplicative: $I(mn) = I(m)I(n)$ for relatively prime m, n.

5. Comparing abundancy indices: Suppose $p_1, p_2, ..., p_k$ are distinct primes, $q_1, q_2, ..., q_k$ are distinct primes, $p_j \leq q_j$ for every $j \in \{1, 2, ..., k\}$, and $e_1, ..., e_k$ are positive integers. Then $I(\prod_{j=1}^{k} p_j^{e_j}) \geq I(\prod_{j=1}^{k} q_j^{e_j})$ with equality iff $p_j = q_j$ for every $j \in \{1, 2, ..., k\}$.

6. Upper bound for abundancy index of a given n: If the distinct prime factors of n are $p_1, p_2, ...p_k$, then $I(n) < \prod_{j=1}^{k} \frac{p_j}{p_j-1}$.

7. If $\gcd(n, \sigma(n)) = 1$, then n is solitary.

   Proof. Let $\gcd(n, \sigma(n)) = 1$ and suppose n has a friend m. Then $\frac{\sigma(n)}{n} = \frac{\sigma(m)}{m}$, so $m\sigma(n) = n\sigma(m)$. Thus $n|m\sigma(n)$, so $n|m$. Therefore, by Property 2, n and m cannot be friends. □

   A corollary of this result is that no prime power has a friend.

2 Friends of $2p$

We will now use the above properties, as well as previous work by Haenel and Wood [1], to establish more specific conditions for the existence of friends of $2p$, where p is a prime greater than 3. Throughout this section, let m denote a positive integer of the form $2p$ for some prime $p > 3$ and suppose n is a friend of m.

Theorem 2.1. $p|n$.

Proof. $\frac{\sigma(m)}{m} = \frac{3}{2}(\frac{p+1}{p}) = \frac{\sigma(n)}{n}$. This implies $2p\sigma(n) = 3n(p+1)$. Since $p|3n(p+1)$ and p is a prime other than 3, it follows that $p|n$. □

Theorem 2.2. $3|\sigma(n)$.

Proof. The above equation also implies $3|2p\sigma(n)$, so $3|\sigma(n)$. □

Theorem 2.3. $2 \not| n$.

Proof. Suppose $2|n$. Then by Theorem 1, $2p|n$. But then by Property 2, $I(n) \geq I(2p)$ with equality iff $n = 2p$. Therefore, $2 \not| n$. □

Theorem 2.4. *n has at least three distinct prime factors. If n has exactly three distinct prime factors, then the smallest of these factors must be 3.*

Proof. By the corollary of Property 7, $n = 2p$ must have at least two distinct prime factors. Haenel and Wood [1] established that n has exactly two distinct prime factors if and only if n is of the form $3^b p^2$ and $3^{b+1} - 1 = p(1+p)$, where $b \in \mathbb{Z}^+$. They acknowledged that they had yet to find any prime p such that $2p$ has a friend satisfying this condition, but they had also been unable to prove that such a prime does not exist.

We will prove that there is no prime p satisfying $3^{b+1} - 1 = p(1+p)$ for

a positive integer b. We must consider three possible cases.
Case 1: $p \equiv 0 \bmod 3$.
This means $p = 3$, but we are requiring $p > 3$.
Case 2: $p \equiv 1 \bmod 3$.
Then $p = 3x + 1$ for some positive integer x.
$3^{b+1} - 1 = (3x + 1)(3x + 2)$
$3^{b+1} = 9x^2 + 9x + 3$
$3^b = 3x^2 + 3x + 1$, which is impossible.
Case 3: $p \equiv 2 \bmod 3$.
Then $p = 3x + 2$ for some positive integer x.
$3^{b+1} - 1 = (3x + 2)(3x + 3)$
3 divides the right-hand side of the above equation, but not the left-hand side. This contradiction implies that $p \not\equiv 2 \bmod 3$.
Therefore, there is no prime p such that $p(1 + p) = 3^{b+1} - 1$. Thus, we can conclude that n must have at least 3 distinct prime factors.

Now suppose n has exactly 3 distinct prime factors x, y, z. Then $n = x^a y^b z^c$ for some $a, b, c \in \mathbb{Z}^+$. Without loss of generality, assume $x < y < z$. By Theorem 3, $2 \nmid n$, so $3 \leq x < y < z$.
We know $I(n) = \frac{3}{2} \cdot \frac{p+1}{p} > \frac{3}{2}$ and by Property 6, $I(n) < \frac{x}{x-1} \cdot \frac{y}{y-1} \cdot \frac{z}{z-1}$.

- Suppose $x \geq 7$. Then by Properties 5 and 6, $I(n) \leq I(7^a 11^b 13^c) < \frac{7}{6} \cdot \frac{11}{10} \cdot \frac{13}{12} < \frac{3}{2}$. Therefore, $x < 7$.
- Suppose $x = 5$ and $y \geq 11$. Then $I(n) \leq I(5^a 11^b 13^c) < \frac{5}{4} \cdot \frac{11}{10} \cdot \frac{13}{12} < \frac{3}{2}$. Therefore, if $x = 5$, then $y = 7$.
- Suppose $x = 5, y = 7$, and $z > 31$. Then $I(n) \leq I(5^a 7^b 37^c) < \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{37}{36} < \frac{3}{2}$. Therefore, if $x = 5$, we have $y = 7$ and $z \in \{11, 13, 17, 19, 23, 29, 31\}$.

Now let $n = 5^a 7^b z^c, a, b, c \in \mathbb{Z}^+, z \in \{11, 13, 17, 19, 23, 29, 31\}$. By Theorem 1, $p|n$, so $p \in \{x, y, z\}$. Ward [3] showed any friend of 10 must have at least 6 distinct prime factors, so $p \neq 5$.
Suppose $p = 7$. Then $I(n) = \frac{3}{2} \cdot \frac{8}{7} = \frac{12}{7}$.
But $I(n) \leq I(5^a 7^b 11^c) < \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{11}{10} < \frac{12}{7}$, so $p \neq 7$. Therefore, $p = z$.

- Suppose $z \in \{11, 13\}$. Then $I(n) = I(2z) \geq \frac{3}{2} \cdot \frac{14}{13} = \frac{21}{13}$.
  But $I(n) \leq I(5^a 7^b 11^c) < \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{11}{10} < \frac{21}{13}$. So $z \notin \{11, 13\}$.
- Suppose $z \in \{17, 19, 23, 29\}$. Then $I(n) = I(2z) \geq \frac{3}{2} \cdot \frac{30}{29} = \frac{45}{29}$.
  But $I(n) \leq I(5^a 7^b 17^c) < \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{17}{16} < \frac{45}{29}$. So $z \notin \{17, 19, 23, 29\}$.

- Suppose $z = 31$. Then $I(n) = I(2z) = \frac{3}{2} \cdot \frac{32}{31} = \frac{48}{31}$.
  But $I(n) = I(5^a 7^b 31^c) < \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{31}{30} < \frac{48}{31}$. So $z \neq 31$.

We know that if $x = 5$, then $y = 7$ and $p = z \leq 31$. But we've shown contradictions for every prime $z \leq 31$. Therefore, $x \neq 5$, so if n has exactly 3 distinct prime factors, then the smallest of these factors must be 3. ☐

The case where $3|n$ presents unique challenges. When all prime factors of n are greater than 3, our strategy has been to employ Properties 5 and 6 to eliminate possible factorizations of n using proof by contradiction. Employing this strategy, we have significantly narrowed down the possible prime factors for n when $3 \not| n$. However, when $3|n$, Property 6 says that $I(n) < \frac{3}{2} \prod_{j=1}^{k} \frac{p_j}{p_j - 1}$, where $p_1, p_2, ...p_k$ are the other distinct prime factors of n. Thus, when $3|n$ Property 6 always gives an upper bound greater than $\frac{3}{2}$ for $I(n)$. Our methods do not work in this case. Therefore, the rest of this paper will focus on two cases: first, the case where $3|n$, and secondly, the case where $3 \not| n$.

Theorem 2.5. *If $3|n$, then p appears with an exponent greater than 1 in the prime factorization of n.*

Proof. Suppose p is a prime greater than 3 and suppose n is a positive integer such that $3|n$ and $I(n) = I(2p)$. Then, by Theorems 1 and 4, $n = 3^a p^b k^c$ for some positive integers a, b, c, k, where $k > 1$ is relatively prime to 3 and p. Thus,
$\frac{3}{2} \cdot \frac{p+1}{p} = \frac{3^{a+1}-1}{3^a(2)} \cdot \frac{p^{b+1}-1}{p^b(p-1)} \cdot \frac{\sigma(k^c)}{k^c}$
Cancelling out the $2p$ term, we get $3(p+1) = \frac{3^{a+1}-1}{3^a} \cdot \frac{p^{b+1}-1}{p^{b-1}(p-1)} \cdot \frac{\sigma(k^c)}{k^c}$
If $b = 1$, then $3(p+1) = \frac{3^{a+1}-1}{3^a} \cdot \frac{p^2-1}{p-1} \cdot \frac{\sigma(k^c)}{k^c} \Rightarrow 3 = \frac{3^{a+1}-1}{3^a} \cdot \frac{\sigma(k^c)}{k^c} \Rightarrow$
$3^{a+1}k^c = (3^{a+1} - 1)\sigma(k^c)$.
By Theorem 3, $2 \not| k^c$. Therefore, the left-hand side of the above equation is odd, while the right-hand side is even. This contradiction implies $b > 1$. ☐

Theorem 2.6. *If $3 \not| n$ and n has exactly four distinct prime factors, then its smallest prime factor is 5, its second smallest prime factor is 7, and its third smallest prime factor is less than or equal to 31.*

Proof. Suppose that p is a prime greater than 3 and suppose n is a friend of $2p$ with exactly four distinct prime factors, where 3 is not one of these factors. (In this case, we can actually specify that $p > 5$ thanks to the work done by Ward [3].) This means $n = w^a x^b y^c z^d$ for distinct primes w, x, y, z and positive integers a, b, c, d (not necessarily distinct). Without loss of generality, assume $w < x < y < z$. By Theorem 3, we know that $w > 2$, which means in this case $w > 3$. We also know that $I(n) = \frac{3}{2} \cdot \frac{1+p}{p} > \frac{3}{2}$.

Suppose $w > 5$. Then by Properties 5 and 6 of the abundancy index, $I(n) \leq I(7^a 11^b 13^c 17^d) < \frac{7}{6} \cdot \frac{11}{10} \cdot \frac{13}{12} \cdot \frac{17}{16} < \frac{3}{2}$. Therefore, $w = 5$ and $p \in \{x, y, z\}$.

Suppose $x > 13$. Then $I(n) \leq I(5^a 17^b 19^c 23^d) < \frac{5}{4} \cdot \frac{17}{16} \cdot \frac{19}{18} \cdot \frac{23}{22} < \frac{3}{2}$. Therefore, $x \in \{7, 11, 13\}$.

Suppose $x \geq 11$, so $y \geq 13$. Then by Property 5, $I(n) \geq I(2z) = \frac{3}{2} \cdot \frac{z+1}{z}$. By Property 6, $I(n) \leq I(5^a 11^b 13^c z^d) < \frac{5}{4} \cdot \frac{11}{10} \cdot \frac{13}{12} \cdot \frac{z}{z-1}$

Thus we have

$\frac{5}{4} \cdot \frac{11}{10} \cdot \frac{13}{12} \cdot \frac{z}{z-1} > \frac{3}{2} \cdot \frac{z+1}{z} \Rightarrow \frac{143}{144} > \frac{z^2-1}{z^2} \Rightarrow z^2 < 144$, contradicting $z > y \geq 13$. Therefore, $x = 7$.

Finally, suppose $x = 7$ and $y > 31$. Then by Property 5, $I(n) \geq I(2z) = \frac{3}{2} \cdot \frac{z+1}{z}$. By Property 6, $I(n) \leq I(5^a 7^b 37^c z^d) < \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{37}{36} \cdot \frac{z}{z-1}$.

It follows that

$$\frac{5}{4} \cdot \frac{7}{6} \cdot \frac{37}{36} \cdot \frac{z}{z-1} > \frac{3}{2} \cdot \frac{z+1}{z} \Rightarrow \frac{1295}{1296} > \frac{z^2-1}{z^2} \Rightarrow z^2 < 1296 \Rightarrow z < 36,$$

a contradiction since $y \geq 37$ implies $z \geq 41$.

Therefore, $y \leq 31$. □

This process of eliminating potential prime factors of n becomes significantly more difficult when $3|n$. We can still eliminate some prime factors, but only when we make assumptions about the exponent of 3 in the prime factorization of n. The next two lemmas are included as an aside to show the challenges of using this method.

Lemma 2.7. *If n has exactly four distinct prime factors, the smallest of which is 3, and if 3 appears with an exponent of 1 in the prime factorization of n, then n's second smallest prime factor is less than or equal to 19.*

Proof. Given the above conditions, $n = 3x^b y^c z^d$ for some primes $3 < x < y < z$ and positive integers b, c, d. Then, by Property 6, $I(n) < \frac{4}{3} \cdot \frac{x}{x-1} \cdot \frac{y}{y-1} \cdot \frac{z}{z-1}$.

If $x \geq 23$, then $I(n) \leq I(3 \cdot 23^b \cdot 29^c \cdot 31^d) < \frac{4}{3} \cdot \frac{23}{22} \cdot \frac{29}{28} \cdot \frac{31}{30} < \frac{3}{2}$.

This contradicts $I(n) = \frac{3}{2} \cdot \frac{p+1}{p} > \frac{3}{2}$. Therefore, $x \leq 19$. □

Lemma 2.8. *If n has exactly four distinct prime factors, the smallest of which is 3, and if 3 appears with an exponent of 2 in the prime factorization of n, then n's second smallest prime factor is less than or equal to 73.*

Proof. Given the above conditions, $n = 9x^b y^c z^d$ for some primes $3 < x < y < z$ and positive integers b, c, d. Then, by Property 6, $I(n) < \frac{13}{9} \cdot \frac{x}{x-1} \cdot \frac{y}{y-1} \cdot \frac{z}{z-1}$. Using the same methods as the first lemma, if $x \geq 79$, then $I(n) \leq I(9 \cdot 79^b \cdot 83^c \cdot 89^d) < \frac{13}{9} \cdot \frac{79}{78} \cdot \frac{83}{82} \cdot \frac{89}{88} < \frac{3}{2}$. Therefore, $x \leq 73$. $\square$

As the exponent on 3 increases, so do the possibilities for the next smallest prime factor of n. It is easy to see why this method needs refining in order to be of any real use in the case that $3|n$, but such a refinement is certainly possible.

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