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Some Bayesian estimators of the reliability inverse Weibull distribution using Doubly Type II Censored data

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Abstract

In this paper, we find a Bayesian estimator for the scale parameter β and the reliability of the inverse Weibull distribution, assuming that there are two prior distributions for the parameter: the first is non-informative (Quasi) whereas the second is informative represented by the (Gamma) distribution. By using three loss functions (Error-square, De-Groot and precautionary function), the estimation was done under Doubly Type II Censored data which was conducted to generate random variables based on the Monte Carlo simulation study to find out the best estimator for the parameter and reliability based on the mean square error.

1 Introduction

In reliability and survival analysis, the Inverse Weibull distribution (IWD) is a crucial life-time model that may be used to simulate a number of failure characteristics. In addition, it can be utilized to calculate the cost efficacy

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and maintenance intervals of reliability-centered maintenance activities [7]. It is crucial for a variety of applications, such as the dynamic parts of diesel engines and the rate at which an insulating fluid degrades when subjected to continual strain [8]. The two parameters are said to follow a random variable x. If the pdf for the inverse Weibull distribution is provided by [1, 3]:

$$f\left(x;\tau,\beta\right)=\tau\beta e^{-\beta x^{-\tau}}x^{-(\tau+1)}\ x>0,\ \tau,\beta> \tag{1.1}$$

where τ,β are the shape and scale parameters respectively, then the cumulative and the reliability functions are given by:

$$F\left(x\right)=e^{-\beta x^{-\tau}} \tag{1.2}$$

$$R\left(x\right)=1-e^{-\beta x^{-\tau}} \tag{1.3}$$

Bayesian analysis was first used in 1763 [10], with the prior distribution g(β) and the model using the unknown parameter (β) as a r.v. The Bayesian considers each data observation's density as a conditional density that depends on the realization of the random variable β with the posterior density [9]:

$$\psi\left(\beta|x\right)=\frac{f(x|\beta)g(\beta)}{\int f(x|\beta)g(\beta)d\beta} \tag{1.4}$$

For the scale parameter (β) and double type II censored data, the Bayesian estimation under two prior functions is as follows [2]:

$$L\left(\beta|x\right)=\left(\frac{n!}{(r-1)!\left(n-s\right)1}\right)\left(F\left(x_r\right)\right)^{r-1}\left(1-F\left(x_i\right)\right)^{n-s}\prod_{i=r}^{s}f(x_i;\beta) \tag{1.5}$$

There is always some discrepancy between the estimate and the parameter, where the loss function $L(\hat{\beta},\beta)$ denotes losses suffered while we estimate [11]. Loss functions come in a variety of forms. Here, we employ three loss functions (squared error, De-Groot, and Precautionary). Now the Bayesian estimator for the scale parameter (β) under doubly type II censored data, by Eq. 1.1, 1.2 and 1.5.

$$L\left(\beta|X\right)=\frac{n!}{(u-1)!\left(n-t\right)!}\prod_{i=u}^{t}\left(\tau\beta e^{-\beta x_i{}^{-\tau}}x_i{}^{-(\tau+1)}\right)\left(e^{-\beta x_i{}^{-\tau}}\right)^{u-1}\left(1-e^{-\beta x_i{}^{-\tau}}\right)^{n-t}$$

where $L\left(\beta|x\right)=\frac{n!}{(u-1)!(n-t)!}k_1\tau^m\beta^m e^{-\beta(\sum_{i=u}^{t}x_i{}^{-\tau}+(u-1)x_i{}^{-\tau})}\left(1-e^{-\beta x_i{}^{-\tau}}\right)^{n-t}$

$$m=t-u+1,\ \ \prod_{i=u}^{t}\left(\tau\beta e^{-\beta x_i{}^{-\tau}}x_i{}^{-(u+1)}\right)=\tau^{t-u+1}\beta^{t-u+1}e^{-\beta\sum_{i=u}^{t}x_i{}^{-\tau}}\prod_{i=u}^{t}x_i{}^{-(\tau+1)}$$

$$= \prod_{i=u}^{t} {x_i}^{-(\tau+1)} = e^{\sum_{i=u}^{t} \ln {x_i}^{-(\tau+1)}} = e^{-(\tau+1)\sum_{i=u}^{t} ln x_i} = k_1$$

We have $(1-x)^n = \sum_{j=0}^{n} C_j^n (-1)^j x^j$ (binomial characteristic), then

$$L(\beta|x) = \frac{n!}{(u-1)!\,(n-t)!} k_1 \tau^m \beta^m e^{-\beta(\sum_{i=u}^{t} {x_i}^{-\tau} + (u-1){x_u}^{-\tau})} \sum_{j=0}^{n-t} C_j^{n-t} (-1)^j e^{-\tau_j x_t{}^{-\tau}}$$

$$L(\beta|x) = \frac{n!}{(u-1)!\,(n-t)!} k_1 \tau^m e^{-\beta k} \beta^m \sum_{j=0}^{n-t} C_j^{n-t} (-1)^j, \qquad (1.6)$$

where $k = \sum_{i=u}^{t} {x_i}^{-\tau} + (u-1){x_u}^{-\tau} + j{x_t}^{-\tau}$

a. Posterior distribution under Quasi prior [7] as:

$$g(\beta) = \frac{1}{\beta^c} \quad \beta, c > 0 \qquad (1.7)$$

by Equations 1.4, 1.5, 1.6 and 1.7, $\int_0^\infty \frac{n!}{(u-1)!(n-t)!} k_1 \tau^m e^{-\beta k_j} \beta^{-c} \beta^m \sum_{j=0}^{n-t} C_j^{n-t} (-1)^j d\beta$

$$= \frac{n!}{(u-1)!\,(n-t)!} k_1 \tau^m \sum_{j=0}^{n-t} C_j^{n-t} (-1)^j \int_0^\infty \beta^{m-c} e^{-\beta k_j} d\beta$$

by $\int_0^\infty x^{a-1} e^{-bx} dx = \frac{\Gamma a}{b^a}$, the posterior function under Quasi prior as:

$$\psi_Q(\beta|x) = \frac{\sum_{j=0}^{n-t} C_j^{n-t} (-1)^j \beta^{m-c} e^{-\beta k_j}}{\sum_{j=0}^{n-t} C_j^{n-t} (-1)^j \frac{\Gamma m-c+1}{{k_j}^{m-c+1}}} \qquad (1.8)$$

Currently, the three loss functions are used by the Bayes estimator for β and reliability function under Quasi-priors as follows:

1. by squared error loss function: $\hat{\beta} = E(\beta \backslash x)$

$$\therefore \hat{\beta}_{s1} = \frac{\sum_{j=0}^{n-t} C_j^{n-t} (-1)^j \frac{\Gamma m-c+2}{{k_j}^{m-c+2}}}{\sum_{j=0}^{n-t} C_j^{n-t} (-1)^j \frac{\Gamma m-c+1}{{k_j}^{m-c+1}}} \qquad (1.9)$$

and the approximate reliability function by Eq. 1.3 as:

$$\hat{R}_{s1} = 1 - e^{-\hat{\beta}_{s1} x^{-\tau}} \qquad (1.10)$$

2. by De-Groot loss function: $\hat{\beta} = \frac{E(\beta^2|x)}{E(\beta|x)}$

$$\therefore \hat{\beta}_{D1} = \frac{\sum_{j=0}^{n-t} C_j^{n-t}(-1)^j \frac{\Gamma m-c+3}{k_j{}^{m-c+3}}}{\sum_{j=0}^{n-t} C_j^{n-t}(-1)^j \frac{\Gamma m-c+2}{k_j{}^{m-c+2}}} \tag{1.11}$$

and the approximate reliability function by Eq. 1.3 as:

$$\hat{R}_{D1} = 1 - e^{-\hat{\beta}_{D1} x^{-\tau}} \tag{1.12}$$

3. by precautionary loss function: $\hat{\beta} = [E(\beta^2 \backslash x)]^{\frac{1}{2}}$

$$\therefore \hat{\beta}_{p1} = \left[\frac{\sum_{j=0}^{n-t} C_j^{n-t}(-1)^j \frac{\Gamma m-c+3}{k_j{}^{m-c+3}}}{\sum_{j=0}^{n-t} C_j^{n-t}(-1)^j \frac{\Gamma m-c+1}{k_j{}^{m-c+1}}} \right]^{\frac{1}{2}} \tag{1.13}$$

and the approximate reliability function by Eq. 1.3 as:

$$\hat{R}_{p1} = 1 - e^{-\hat{\beta}_{p1} x^{-\tau}} \tag{1.14}$$

b. The posterior function under Gamma prior [1] as:

$$g(\beta) = \frac{\vartheta^\delta}{\Gamma\delta} \beta^{\delta-1} e^{\vartheta\beta} \quad \beta > 0 \tag{1.15}$$

By Equations 1.6 and 1.15, we get:

$$\int_0^\infty L(\beta|x)\, g(\beta)\, d\beta = \frac{n! k_1 \tau^m}{(u-1)!\,(n-t)!} \sum_{j=0}^{n-t} C_j^{n-t}(-1)^j e^{-\beta k_j} \beta^m \frac{\vartheta^\delta}{\Gamma\delta} \beta^{\delta-1} e^{\vartheta\beta} d\beta$$

the posterior function under Gamma prior as:

$$\psi_G(\beta|x) = \frac{\sum_{j=0}^{n-t} C_j^{n-t}(-1)^j \beta^{m+\delta-1} e^{-(\vartheta+k_j)\beta}}{\sum_{j=0}^{n-t} C_j^{n-t}(-1)^j \frac{\Gamma m+\delta}{(\vartheta+k_j)^{m+\delta}}} \tag{1.16}$$

Currently, the three loss functions are used by the Bayes estimator for β the reliability function and under Gamma-prior as follows:

1. by squared error loss function: $\hat{\beta} = E(\beta|x)$

$$\therefore \hat{\beta}_{s2} = \frac{\sum_{j=0}^{n-t} C_j^{n-t}(-1)^j \frac{\Gamma m+\delta+1}{(\vartheta+k_j)^{m+\delta+1}}}{\sum_{j=0}^{n-t} C_j^{n-t}(-1)^j \frac{\Gamma m+\delta}{(\vartheta+k_j)^{m+\delta}}} \tag{1.17}$$

and the approximate reliability function by Eq. 1.3 as:

$$\hat{R}_{s2} = 1 - e^{-\hat{\beta}_{s2} x^{-\tau}} \tag{1.18}$$

2. by De-Groot loss function : $\hat{\beta} = \frac{E(\beta^2|x)}{E(\beta|x)}$

$$\therefore \hat{\beta}_{D2} = \frac{\sum_{j=0}^{n-t} C_j^{n-t}(-1)^j \frac{\Gamma m+\delta+2}{(\vartheta+k_j)^{m+\delta+2}}}{\sum_{j=0}^{n-t} C_j^{n-t}(-1)^j \frac{\Gamma m+\delta+1}{(\vartheta+k_j)^{m+\delta+1}}} \tag{1.19}$$

and the approximate reliability function by Eq. 1.3 as:

$$\hat{R}_{D2} = 1 - e^{-\hat{\beta}_{D2}x^{-\tau}} \tag{1.20}$$

3. by precautionary loss function: $\hat{\beta} = [E(\beta^2\backslash x)]^{\frac{1}{2}}$

$$\therefore \hat{\beta}_{p2} = \left[\frac{\sum_{j=0}^{n-t} C_j^{n-t}(-1)^j \frac{\Gamma m+\delta+2}{(\vartheta+k_j)^{m+\delta+2}}}{\sum_{j=0}^{n-t} C_j^{n-t}(-1)^j \frac{\Gamma m+\delta}{(\vartheta+k_j)^{m+\delta}}}\right]^{\frac{1}{2}} \tag{1.21}$$

and the approximate reliability function by Eq. 1.3 as:

$$\hat{R}_{p2} = 1 - e^{-\hat{\beta}_{p2}x^{-\tau}} \tag{1.22}$$

2 Simulation outcomes

By utilizing the MSE as a measure of precision to compare each estimate's performance, the Monte Carlo simulation study was performed to compare the best Bayes estimator for scale parameter and reliability function for IWD under two prior functions. In this simulation study , generated values for r.v.'s of size $(n = 10, 20, 30, 50)(u = 4, 8, 15, 25), (t = 8, 16, 32, 42)$ with default values scale parameter $(\beta = 1.5, 3, 5)$ and the parameters values of prior distributions $[\tau = 0.9, c = 2, \delta = 3, \vartheta = 0.8], L = 500$. The following tables contain a summary and tabulated results:

$$MSE\left(\hat{\beta}\right) = \frac{\sum_{i=1}^{L}\left(\hat{\beta}-\beta\right)^2}{L}, \; and \quad MSE\left(\hat{R}\right) = \frac{\sum_{i=1}^{L}\left(\hat{R}-R\right)^2}{L}$$

Table 1: Mean and MSE for $\hat{\beta}$ and $\hat{R}$ with β =1.5

n	S1		D1	P1	S2	D2	P2
10	Mean	1.4982	1.6859	1.5893	1.9268	2.0878	2.0057
	$MSE_{\hat{\beta}}$	0.3101	0.4272	0.3569	0.5330	0.7573	0.6358
	$MSE_{\hat{R}}$	0.2231	0.1111	0.1620	0.2604	0.0769	0.1444
20	Mean	1.5032	1.5973	1.5495	1.7412	1.8284	1.7843
	$MSE_{\hat{\beta}}$	0.1371	0.1642	0.1481	0.2144	0.2801	0.2448
	$MSE_{\hat{R}}$	0.1080	0.0591	0.0811	0.1191	0.0478	0.0764
30	Mean	1.4943	1.5566	1.5251	1.6580	1.7173	1.6874
	$MSE_{\hat{\beta}}$	0.0848	0.0952	0.0890	0.1187	0.1478	0.1322
	$MSE_{\hat{R}}$	0.0744	0.0401	0.0552	0.0796	0.0345	0.0529
50	Mean	1.5045	1.5422	1.5232	1.6055	1.6420	1.6236
	$MSE_{\hat{\beta}}$	0.0632	0.0681	0.0653	0.0786	0.0908	0.0843
	$MSE_{\hat{R}}$	0.0523	0.0244	0.0360	0.0543	0.0223	0.0350

Table 2: Mean and MSE for $\hat{\beta}$ and $\hat{R}$ with β =3

n	S1		D1	P1	S2	D2	P2
10	Mean	2.9642	3.3356	3.1445	3.3603	3.6409	3.4978
	$MSE_{\hat{\beta}}$	1.1138	1.5213	1.2727	0.8863	1.2986	1.0673
	$MSE_{\hat{R}}$	1.0902	0.1112	0.3584	0.9748	0.0770	0.2794
20	Mean	3.0253	3.2147	3.1186	3.2607	3.4240	3.3413
	$MSE_{\hat{\beta}}$	0.6115	0.7358	0.6632	0.5868	0.7519	0.6614
	$MSE_{\hat{R}}$	0.5396	0.0589	0.1811	0.5130	0.0477	0.1584
30	Mean	3.0189	3.1448	3.0812	3.1890	3.3031	3.2456
	$MSE_{\hat{\beta}}$	0.3889	0.4426	0.4114	0.3874	0.4692	0.4246
	$MSE_{\hat{R}}$	0.3567	0.0401	0.1209	0.3457	0.0346	0.1103
50	Mean	2.9815	3.0561	3.0186	3.0907	3.1611	3.1257
	$MSE_{\hat{\beta}}$	0.2423	0.2547	0.2484	0.237	0.2659	0.2504
	$MSE_{\hat{R}}$	0.2383	0.0244	0.0768	0.2325	0.0222	0.0723

Table 3: Mean and MSE for $\hat{\beta}$ and $\hat{R}$ with β =5

n	S1		D1	P1	S2	D2	P2
10	Mean	4.9720	5.5950	5.2743	4.8420	5.2461	5.0400
	$MSE_{\hat{\beta}}$	3.6361	4.9574	4.1661	1.2787	1.5319	1.3598
	$MSE_{\hat{R}}$	1.7739	0.1112	0.5719	0.9226	0.0770	0.3924
20	Mean	5.0267	5.3414	5.1817	4.9669	5.2156	5.0897
	$MSE_{\hat{\beta}}$	0.1371	0.1642	0.1481	0.2144	0.2801	0.2448
	$MSE_{\hat{R}}$	1.3966	0.0589	0.2912	1.1421	0.0477	0.2362
30	Mean	5.0013	5.2100	5.1046	4.9738	5.1517	5.0620
	$MSE_{\hat{\beta}}$	1.1370	1.2770	1.1953	0.7997	0.8802	0.8314
	$MSE_{\hat{R}}$	1.0627	0.0400	0.2083	0.9083	0.0345	0.1785
50	Mean	5.0008	5.1267	5.0634	4.9901	5.1038	5.0467
	$MSE_{\hat{\beta}}$	0.6533	0.7022	0.6735	0.5319	0.5671	0.5461
	$MSE_{\hat{R}}$	0.6233	0.0244	0.1241	0.5668	0.0222	0.1129

3 Analysis of results

In Table 1 and for $\beta = 1.5$, we see the best estimator of the parameter with square error loss function under Quasi prior and for all sample size whereas in Table 2 when $\beta = 3$ also the best estimator with square error loss function but under Gamma prior for all sample size except in $n = 10$ the best with precautionary under the same prior. In the last Table 3 for $\beta = 5$, the best estimator with square error loss function under Gamma prior and for all sample size whereas for the Reliability function estimators and for all sample size ($n = 10, 20, 30, 50$) the best is under Gamma prior using De-Groot loss function.

References

[1] A. H. Abd Ellah, Bayesian and non- Bayesian estimation of the inverse Weibull model based on generalized order statistics, Intelligent information Management, **4,** (2012), 23–31.

[2] A. Kasim, Bayes estimators of the shape parameter of Exponentiated Rayleigh distribution, Thesis, Mustansiryah University, (2014).

[3] R. Calabria, G. Pulcini, Bayes 2-sample prediction for the Inverse Weibull distribution, Communications in Statistics, **23,** no. 6, (1994), 1811–1824.

[4] R. Calabria, G. Pulcini, On the maximum likelihood and least squares estimation in Inverse Weibull distribution, Statistics Applicata, **2,** (1990), 53–66.

[5] J. Xiong, Weighted Inverse Weibull and Beta Inverse Weibull, Thesis, Georgia Southern university, (2010), https://digitalcommons.georgiasouthern.edu/etd/661.

[6] K. S. Sultan, N. H. Alsadat, D. Kundu, Bayesian and maximum likelihood estimations of the inverse Weibull parameters under progressive type-II censoring, Journal of Statistical Computation and Simulation, **84,** no. 10, (2014), 2248–2265.

[7] M. A. T. Elshahat, A. E. Ismail, Quasi- Bayesian Estimation Inverse Weibull distribution, International Journal of Mathematics and Statistics Studies, **2,** no. 2, (2014), 64–75.

[8] G. W. Mavis Pararai, A new class of Generalized IWD with applications, Journal of applied Mathematics, **4,** no. 2, (2014), 17–35.

[9] Q. M. Abudl-Sattar, A. R. Al-Dubaicy, The Exponentiated Gumbel Bayesian Estimation under Singly type II Data, Journal of Physics: IOP, **37,** no. 1, (2022), 237.

[10] J. G. Surles, W. J. Padgett, Inference for reliability and stress-strength for a scale Burr type X distribution, Lifetime Data Analysis, **7,** (2001), 187–200.

[11] A. Zellner, An Introduction to Bayesian Inference in Econometrics, Wiley, New York, 1971.