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Walrasian Auctioneer: An Application of Brouwer's Fixed Point Theorem

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Abstract

One of this paper's main goals is to demonstrate Brouwer's fixed point theorem which mainly depends on the closed unit ball fixed point condition in $\mathbb{R}^n$. After that, we demonstrate the Walrasian auctioneer which is an economical application of Brouwer's fixed point theorem where it is a hypothetical technique of price adjustment that mimics how markets reach equilibrium as it increases the price of a good if demand exceeds supply. After price has increased, the Walrasian normalizes the increase. When this new pricing is revealed, the process is then repeated.

1 Introduction

In terms of simulating a decentralized economy, the Arrow-Debreu general equilibrium model is the gold standard. According to this paradigm, each

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person has an endowment (a certain amount of commodities that are his/her own) and a certain set of desires (represented by a utility function). Since people are totally independent, they are all trying to maximize their own usefulness [2]. Yet there is a point, the competitive equilibrium, that, within the limits of the economy, maximizes the utility of all individuals with only a few fundamental assumptions about preferences (the sum of all endowments). Despite the fact that the people in this model solely care about advancing their own interests and have no concern for those of others, the competitive equilibrium ensures that the outcome is the best possible one for everyone in society. We must first prove Brouwer's fixed point theorem, one of the most significant fixed point theorems, in order to provide this existence proof. We must first demonstrate a more fundamental conclusion; namely, that the closed unit ball in $\mathbb{R}^n$ has the fixed point property, in order to prove Brouwer's fixed point theorem. We shall demonstrate Brouwer's fixed point theorem [3] using this finding together with three more theorems. Following that, we'll formally introduce the Arrow-Debreu model for an economy with L people and N distinct products to start the existence proof. The price adjustment process in a competitive market may then be represented by a function once certain fundamental assumptions have been made [7]. We shall demonstrate that this function has a set of prices that defines a fixed point and that it meets the requirements of Brouwer's fixed point theorem. Finally, we shall demonstrate that competitive equilibrium is attained with these prices.

2 Preliminaries

For many areas of mathematical analysis and its applications, fixed point theory is a crucial tool. Confirming the presence of a solution to a set of equations is one of the fixed point theorems' main applications. Roughly speaking, there are three basic techniques in this theory: the metric, the topological, and the order-theoretic approach, with the Banach's, Brouwer's, and Tarski's theorems, respectively, serving as exemplary instances of each. For any space X and any continuous function $f : X \to X$, a fixed point $x \in X$ satisfies $f(x) = x$. Two basic theorems regarding fixed points are Brouwer and Banach theorems and they illustrate the difference between two main branches of fixed point theory which are topological fixed point theory and metric space fixed point theory.

For Brouwer theorem, X is a closed unit ball in the Euclidean space. So any function must have a fixed point but the set of all fixed points need not be a one-point set. The core of Brouwer theorem is the compactness and contactibility of the unit ball. For Banach theorem, if f is a contraction and X is a complete metric space where a metric on X is used in the conclusive assumption that f is a contraction, then it must have a unique fixed point. In the Euclidean space X, the metric topology specifies continuity of functions and the unit ball is considered a metric space. Brouwer's fixed point theorem was initiated and became a fundamental branch of the general topology. Homology theory was used by Lipschitz in order to study fixed point theory and Brouwer's generalized results successfully. Moreover, Lipschitz introduced the number $L(f)$ of the continuous functions $f : (X, d) \rightarrow (X, \rho)$ where f induces the homomorphism $f^* : H_*(X; Q) \rightarrow H_*(X; Q)$ on the homology groups of the space X. Now, $L(f)$ is defined as follows: $L(f) = \sum_k (-1)^k tr\{f^* : H_*(X; Q) \rightarrow H_*(X; Q)\}$, it is defined by the homology homomorphism and it is a homotopy invariant. In the homotopy classes, Nielsen tried to find the minimal number of fixed points.

For Nielsen's number, if x and y are two fixed points in the topological space (X, τ), a function $f : X \rightarrow X$ and ω is the path relating x and y, the two paths ω and $f \circ \omega$ are homotopic relative to the end points. On the set of all fixed points of the function f, an equivalence relation is defined and the equivalence classes are fixed point classes. The set of all fixed points is the union of fixed point classes in X. If the fixed point class has finitely many fixed points, then its index at each point of the fixed point class equals the sum of the winding numbers. The essential fixed point class is a class that has a nonzero index and the number of essential fixed point classes is the Nielsen's number $N(f)$ [1]. If $\widetilde{X}$ is the universal covering space of X and the group of all covering transformations is denoted by G, then the group homeomorphism $h : G \rightarrow G$ given by $h(a)\widetilde{f} = \widetilde{f}a$ $\forall a \in G$ and a fixed lifting function $\widetilde{f} : \widetilde{X} \rightarrow \widetilde{X}$. Now, h defines an equivalence on the group G; that is, if a and b are two elements in G, $a \sim bah(b)^{-1}$. Reidemeister classes $[a]$ are equivalence classes and G_h is the set of all Reidemeister classes. Typically, a one-to one correspondence between the Reidemeister classes and the Nielsen's fixed point classes is satisfied. The indexing of Nielsen's classes F is by Reidemeister's classes $[a]$ such that $F = F_a$. The number of the Reidemeister classes is Reidemeister's number and denoted by $R(h)$. The Lebschetz $L(f) = \sum_{[a]} index(f, F_a)$ is an integer and equals the non-zero terms of Reidemeister trace $R(f) = \sum_{[a]} index(f, F_a)$. Wecken showed that manifolds with dimension greater than or equal to 3, $N(f)$ will be the minimal number of fixed points in the homotopy class. Schirmer showed that in manifolds with dimension greater than or equal to 3 in the homotopy class, Nielsen's coincidence number will be the minimal number of coincidences and this result generalizes Wecken's result from fixed points to coincidences. (Co)homology theories are useful tools solving fixed point theory problems. If f and g are two continuous functions such that $f, g : M \to N$, where M and N are closed orientable manifolds with equal dimension n, then $f_* : H_q(M, \mathbb{Q}) \to H_q(N, \mathbb{Q})$ is equivalent to $g_* : H_q^{n-q}(N, \mathbb{Q}) \to H_q^{n-q}(M, \mathbb{Q})$ and $\mu : H_q^{n-q}(M, \mathbb{Q}) \to H_q(M, \mathbb{Q})$ is equivalent to $\nu : H_q^{n-q}(N, \mathbb{Q}) \to H_q(N, \mathbb{Q})$. Considering the composite $\mu g^* v^{-1} f_* : H_q(M, \mathbb{Q}) \to H_q(M, \mathbb{Q})$, $L(f, g) = \sum_{n=1}^{n} (-1)^q tr(\mu g^* v^{-1} f_*)$. If $L(f, g) \neq 0$, then $\exists x$ a coincidence point: $f(x) = g(x)$.

3 Brouwer's Fixed Point Theorem for Unit Ball

The unit ball B_n has a fixed point property for $n = 1, 2, ...$ and the topological space (X, τ) has the fixed point property if $\forall f : X \to X$ has a fixed point provided that f is continuous.

3.1 Brouwer's Fixed point Theorem

If A be a nonempty closed bounded and convex subset of $\mathbb{R}^n$ for $n = 1, 2, ...$ and $f : A \to A$ is a continuous, then $\exists x \in X$: $f(x) = x$.
Brouwer's fixed point theorem is used to show the existence of solutions of a system of equations provided that f is a continuous self mapping from a compact convex set X.

Theorem 3.2 If $f : B_n \to \mathbb{R}^n$ is a continuous function such that $f(A^{n-1}) \subset B_n$ $\forall n = 1, 2, ...$, then f has a fixed point.

Proof: Suppose that $f : B_n \to \mathbb{R}^n$ and let $g(x) = x$ if $x \in B_n$ and $g(x) = \frac{x}{d_2(x,0)}$ otherwise. Then $g \circ f : B_n \to B_n$ and $\exists x \in B_n$ such that $g(f(x)) = x$.

Kakutani's theorem [8] generalized Brouwer theorem, where if X is a non-empty closed bounded and convex subset of $\mathbb{R}$ and Γ is a valued self correspondence convex set with a closed graph, then $\exists x \in X$ such that $x \in \Gamma(x)$.

Corollary 3.3 (Iterations and error bounds): If $f : X \to X$ is an iterative sequence $f(x_n) = x_{n+1}$ with $x_0 \in X$ converges to the unique fixed point in f. Prior estimates are the error estimates $d(x_k, x) \leq \frac{m^k}{1-m} d(x_0, x_1)$. The posterior estimate is $d(x_k, x) \leq \frac{m}{1-m} d(x_{k-1}, x_k)$.

Definition 3.4 Let $A \subset \mathbb{R}^n$ and consider the continuous function $f : A \to \mathbb{R}^n$. Then

i. f is a class of C^1, f has a continuous extension to U (an open neighborhood of A) on which it is continuously differentiable.

ii. f is non-vanishing if $\forall x \in A$, $f \neq 0$.

iii. f is normed if $\forall x \in X$, $\|f(x)\| = 1$

iv. f is tangent to S^{n-1} if $\forall x \in S^{n-1}$, $\langle x, f(x) \rangle = 0$

Theorem 3.5 Let A be a compact subset of $\mathbb{R}^n$ and let $f : A \to \mathbb{R}^n$ be a function of class C^1 on A. Then $\forall x, y \in A$, $\|f(x) - f(y)\| \leq L \|x - y\|$ for some non-negative real number L.

Corollary 3.6: Let $f : S^{n-1} \to R^n$ be a normed vector field of the class C^1 which is tangent to S^{n-1}. Then for some positive real number m and $f_m(x) = x + mF(x)$, we have $f_m(S^{n-1})\sqrt{1 + m^2} S^{n-1}$.

Theorem 3.7 If (X, τ) and (Y, σ) are homeomorphic topological spaces and X has the fixed point property, then Y has the fixed point property.

Proof: Let $f : X \to Y$ be a homeomorphism and $g : Y \to Y$ be a continuous function. Let $h : X \to X$ be given by $h(x) = f^{-1}(g(f(x))$. Now, h is continuous since it is a composition of continuous functions and $\exists x_0 \in X$: $f^{-1}(g(f(x_0)) = x_0$ since X has a fixed point. Hence $f(f^{-1}(g(f(x_0))) = f(x_0)$ which implies that $g(f(x_0)) = f(x_0)$ and $f(x_0) \in Y$. Therefore, Y has the fixed point property.

Definition 3.8 If A is a subset of the topological space (X, τ), then A is a retract if $\exists r : X \to A$ continuous function such that $\forall a \in A$ we have $r(a) = a$. The continuous function r is a retraction.

Theorem 3.9 If the topological space (X, τ) has the fixed point property and the subset A of X is a retract, then A has the fixed point property.

Proof: Suppose that $r : X \to A$ is a retraction and let $f : X \to X$ be a continuous function. If $h : X \to A$ is given by $h(x) = f(r(x))$ $\forall x \in X$, then h is continuous. Since X has the fixed point property and h can be extended to $H : X \to X$, $\forall x_0 \in X$ such that $f(r(x_0)) = x_0$. But $f(r(x_0)) \in A$. Hence $x_0 \in A$ and $r(x_0) = x_0$ since r is a retraction. Thus $f(r(x_0)) = f(x_0) = x_0$. That is, f has a fixed point.

4 Applications of Brouwer's fixed point theorem in economy

At first glance, the Brouwer fixed-point theorem may appear to have nothing to do with any practical use, much alone an economic one. The proof of the presence of competitive equilibrium in a market with L consumers and N items. Before providing the evidence, we first establish the economic model that was employed and any relevant definitions. An individual i is bestowed with a set amount of each good in an economy with N goods denoted by $m^i_1, m^i_2, ..., m^i_N$ where $m^i_k \geq 0\ \forall k = 1, ..., N$.

Definition 4.1
i. The collection of goods $(x^i_1, x^i_2, ..., x^i_N)$ is a bundle [4] where x^i_k is the quantity of good $x_k\ \forall k = 1, 2, ..., N$ that individual i has.
ii. Utility function $U_i(x^i_1, x^i_2, ..., x^i_N)$ is a function that associates a numerical value in utils to a particular bundle.

Remark 4.2 Budget constraint restricts individuals and is given by $\sum_{k=1}^{N} p_k x^i_k = \sum_{k=1}^{N} p_k \omega^i_k ...(1)$.

Individuals convert their endowment to nominal income by selling their endowed items at market prices $(p_1, p_2, ..., p_N)$ in order to consume any bundle outside their endowment. Individuals then acquire the bundle at market pricing that optimizes their utility based on their choices. As a result, people are only allowed to spend up to their endowment's nominal worth. In this paradigm, preferences are taken to be monotonic: greater is better. We assume the person spends 100monotonicity, which leads to the equality (1). We resolve the utility maximization issue for individual i in order to identify the "ideal" bundle given a specific endowment (in essence a constrained optimization problem). This results in a set of demand functions x^i_j for each good j that rely on market prices and the nominal income of the individual: $x^i_j(p_1, p_2, ..., p_N, \sum_{k=1}^{N} p_k \omega^i_k) ...(2)$.

We see the demand functions as being exclusively reliant on price since individual i's endowment is constant. Given a set of market prices, the amount of good $x^i_j(p_1, p_2, ..., p_N)$ that maximizes individual i's utility. In order for there to be market equilibrium, person i must maximize his/her utility, which is precisely x_j. Although demand functions are formed from an individual's preferences (utility function), we assume them as given in order to simplify the model.

Definition 4.3 A market in which a person has no impact on the market price is said to be completely competitive [5]. In a market economy with L persons that is completely competitive, $\sum_{i=1}^{L} \omega_j^i$ provides the supply of each good, x_j. Consider a farmer as an analogy: his crop production is his endowment, and the overall amount of food cannot be more than the sum of all farmers' crop yields. As a result, at equilibrium, the supply of good x_j limits the overall demand for it $\sum_{i=1}^{L} x_j^i \leq \sum_{i=1}^{L} \omega_j^i$...(3)

We establish a new slack variable, s_j, which indicates the amount of good x_j not consumed in order to ensure that (3) holds with equality $\sum_{i=1}^{L} x_j^i + s_j \leq \sum_{i=1}^{L} \omega_j^i$...(4)

We now represent market pricing, consumption bundles, and the slack variables as vectors. The market price (resp. quantity consumed, excess supply) of the good x_j is given by each entry p_j (resp. x_j, s_j) of the price vector $p \in \mathbb{R}^n$ (resp. consumption vector c_i, slack vector s). If each entry $v_j \geq 0$, then $v \geq 0$, if each $v_j \leq 0$, then $v \leq 0$ and if each $v_j = 0$, then $v = 0$. Now, since x_j is supposed to be perfect, we have $p \geq 0$. Because x_j is wanted, the individuals must pay for it.

Typically, rewriting (5) gives $\sum_{i=1}^{L}(x_j^i - \omega_j^i) = -s_j$...(5).

$\sum_{i=1}^{L}(x_j^i - \omega_j^i)$ represents $x_j's$ overall surplus demand as a function of price. Hence $z(p)$ in $\mathbb{R}^n$ is the excess demand vector defined as the collection of excess demand functions for each good x_j. By (3) and (4), $z(p^E) = -s^E$ is non-positive where p^E is the equilibrium price and s^E is the slack vector. By (4) and the factorization of p_k^E, we get $\sum_{k=1}^{N} p_k^E(x_k^i - \omega_k^i) = 0 \ \forall i$ ...(6)

Summing across all L individuals gives $\sum_{i=1}^{L}(\sum_{k=1}^{N} p_k^E(x_k^i - \omega_k^i)) = 0$...(7)

Rearranging terms gives $\sum_{i=1}^{L}(\sum_{k=1}^{N} p_k^E(x_k^i - \omega_k^i)) = p^E.(-s^E)$...(8)

Combining equations (7) and (8) yields $p^E.(-s^E) = 0$

Definition 4.4 In an L individual endowment economy with N goods, a competitive equilibrium is a price vector $p^E \geq 0$ and an allocation (consumption vectors) $\{c_1^E, c_2^E, ..., c_L^E\}$ satisfies the following conditions:

i. The utility maximization issue is resolved by the consumption vectors $\{c_1^E, c_2^E, ..., c_L^E\}$ $\forall i = 1, 2, ..., L$.

ii. $s^E \geq 0$ and $p^E.(-s^E) = 0$.

iii. $z(p^E) = -s^E$ i.e, markets clear where $z(p)$ is a function defined $\forall p \geq 0$.

When we combine i and iii, we get $z(p^E) \leq 0$ and $p^E.z(p^E) = 0$.

Theorem 4.5 Let $z(p)$ be a homogeneous continuous function defined $\forall p > 0$, with the following conditions hold:

i. $z(p)$ is of degree 0 with respect to p.
ii. $p.z(p) = 0$ means that individuals consume their entire income.
Then the price vector $p^E \geq 0$ that represents a competitive equilibrium exists.

Since $z(p)$ is homogeneous and the degree (with respect to p) of z is 0, then without affecting z, we can scale p by any constant. Let p represent the $N-$good market's pricing vector. We construct a new normalized price vector called $p^{'}$, and its k^{th} element is provided by $p^{'}_j = \frac{p_j}{\sum_{k=1}^{N} p_k}$

Definition 4.6 The standard simplex in $\mathbb{R}^n$ is $\Delta^{N-1} = \{\sum_{k=1}^{N} q_k : q \in \mathbb{R}^n$ and $q \geq 0\}$ [6]. Since there are no negative prices, $p^{'}$ lies in the simplex.

Now the simplex is normed and bounded since $\|q\| = \sum_{k=1}^{L} q_k^2 \leq \left(\sum_{k=1}^{L} q_k\right)^2 = 1$. Moreover, the simplex is closed and convex.

5 Contraction Theorem

The Contraction Theorem, also known as Banach's Fixed Point Theorem, refers to certain mappings (sometimes referred to as contractions) of a whole metric space into itself. It specifies requirements necessary for a fixed point's existence and uniqueness, which we shall see is a point that is mapped into itself. The theorem also specifies an iterative procedure by which we can acquire error bounds and approximations to the fixed point.

An $x \in X$ that is mapped onto itself (that is, $f(x) = x$) is a fixed point of a continuous function $f : X \to X$ of a set X onto itself. For example, a translation function $f : X \to X + a$ has no fixed points in $\mathbb{R}$ where $a \in \mathbb{R}$ while a rotation function of the plane has the center of rotation as a fixed point. a fixed point in Banach for the fixed points of certain mappings, the theorem is an existence and uniqueness theorem. We shall see from the proof that it also gives us a constructive method for obtaining ever-improving fixed point approximation, this process is known as iteration. We begin by selecting an arbitrary value for x_0 from a predefined set and then compute a sequence of values for $x_1, x_2, ...$by assuming $f(x_n) = x_{n+1}$ for $n = 0, 1, 2, ...$ Almost all branches of practical mathematics use repetition processes like this, particularly Banach's Fixed Point Theorem often ensures that the scheme will converge and that the result will be unique.

If (X, d) is a metric space, then A continuous function $f : (X, d) \to (X, \rho)$ is a contraction on X if $\exists! m < 1$ a positive constant such that $\forall x, y \in X$, we have $d(f(x), f(y)) \leq md(x, y)$. when interpreting this geometrically, this means $f(x)$ and $f(y)$ are closer than x and y. The constant m is the contraction coefficient.

6 Banach Fixed Point Theorem

If (X, d) is a complete metric space and if $f : (X, d) \to (X, \rho)$ is a contraction on X, then there is a unique $x \in X$ such that $f(x) = x$.

Proof: Let $x_0 \in X$ and $f(x_n) = x_{n+1}$ for $n = 0, 1, 2, ...$

Since there is a unique positive constant $m < 1$ such that $\forall x, y \in X$, $d(f(x), f(y)) \leq md(x, y)$ and $f(x_n) = x_{n+1}$ for $n = 0, 1, 2, ...$ we have:

$d(x_{k+1}, x_k) = d(f(x_k), f(x_{k-1}))$
$\leq md(x_k, x_{k-1}) = md(f(x_{k-1}), f(x_{k-2}))$
$\leq m^2 d(x_{k-1}, x_{k-2})$
$\preceq m^2 d(x_1, x_0)$ for $n > k$.

Thus by the triangle's inequality,

$d(x_k, x_n) \leq d(x_k, x_{k+1}) + d(x_{k+1}, x_{k+2}) + \cdots + d(x_{n-1}, x_n)$
$\leq (m^k + m^{k+1} + \cdots + m^{n-1}) d(x_1, x_0)$
$= m^k \left(\frac{1 - m^{n-k}}{1-k} \right) d(x_1, x_0)$.

Now $m \in (0, 1)$. Hence $1 - m^{n-k} < 1$ and so $d(x_k, x_n) \leq \left(\frac{m^n}{1-k} \right) d(x_1, x_0)$ where $d(x_1, x_0)$ is fixed. making $d(x_k, x_n)$ as small by choosing k sufficiently large. Thus $(x_n)_{n=0}^{\infty}$ is a Cauchy sequence. Now (X, d) is complete. So there exists $x \in X$ such that $x_n \to x$. We claim that x is a fixed point by the triangle's inequality

$d(x, f(x)) \leq d(x, x_k) + d(x_k, f(x))$
$= d(x, x_k) + d(f(x_{k-1}), f(x))$
$\leq d(x, x_k) + md(x_{k-1}, x)$, making this distance very small by choosing k sufficiently large and since $x_n \to x$, we have $d(x, f(x)) = 0 \Rightarrow f(x) = x$. Thus x is a fixed point of f. Let $x \neq y \in X$ such that $x = f(x)$ and $y = f(y)$. Since $d(x, y) = d(f(x), f(y)) \leq md(x, y)$ for some constant $m \in (0, 1)$, we have a contradiction. Hence $d(x, y) = 0$. Consequently, we get uniqueness.

One of the Banach fixed point theorem's key theoretical applications is the demonstration of the existence and uniqueness of differential equations' solutions with adequate regularity. Assuming that the complete metric space A is the set of functions and the function (operator) F is the transformation, we claim that the solution of the differential equation, if it exists, is the fixed point of the function F. Considering the case of

$$y^{'} = e^{-x^2}................(9),$$

the solution is obtained by integrating both sides of (9).

The following theorem generalizes Banach fixed point theorem.

7 Caristi Fixed Point Theorem

Let (X, d) be a complete metric space and let $f : X \to X$ be a function such that $\forall x \in X$, $\exists \varphi$ a lower semicontinuous function bounded from below in $\mathbb{R}^X : d(x, f(x)) \leq \varphi(x) - \varphi(f(x))$. Then f has a fixed point in X [9].

In particular, if $f \in X^X$ is a contraction with $m \in (0, 1)$ a contraction coefficient, then the Caristi's theorem is fulfilled for $f \in \mathbb{R}_+^X$ which is defined as $\varphi(x) = \frac{1}{1-m} d(x, f(x))$ and $\varphi(x) - \varphi(f(x)) > d(x, f(x))$.

For the problem $\max_{(x_n)} f(x_0, x_1) + \sum_{n\in\mathbb{N}} m^n f(x_n, x_{n+1})$ for the continuous bounded function f , if X is a non-empty convex subset of the Euclidean space, then the problem has a unique solution for every initial value.

8 Conclusion

The Walrasian auctioneer is a (hypothetical) method of price adjustment that simulates how markets come to equilibrium. It first declares a series of prices determined by $q \in \Delta^{N-1}$, where $q > 0$. Each individual i responds by listing the quantities $\{x_1^i, x_2^i, ..., x_N^i\}$ they want to buy at those costs. Then, for each good x_k, the Walrasian auctioneer examines the excess demand $z_k(q)$. It reacts by raising the price of good x_k if demand outpaces supply which is shown by $z_k(q) > 0$. The Walrasian normalizes the increasing price of x_k after it has grown in price. The procedure is then repeated when this new pricing is disclosed.

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