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On the Diophantine equation $(p+2)^x - p^y = z^2$, where p is prime and $p \equiv 5 \pmod{24}$

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Abstract

In this paper, we show that $(0, 0, 0)$ is the unique non-negative integer solution (x, y, z) for the Diophantine equation $(p+2)^x - p^y = z^2$, where p is prime and $p \equiv 5 \pmod{24}$.

1 Introduction

In 2013, Sroysang [4] proved that the Diophantine equation $5^x + 7^y = z^2$ has no non-negative integer solution. In 2020, Gupta el at. [1] found all non-negative integer solutions of the Diophantine equation $p^x + (p+2)^y = z^2$, where p and $p+2$ are primes. In 2021, Thongnak el at. [5] proved that the Diophantine equation $7^x - 5^y = z^2$ has the unique non-negative integer solution $(x, y, z) = (0, 0, 0)$. In this paper, we will study all non-negative integer solutions of the Diophantine equation $(p+2)^x - p^y = z^2$, where p is prime (but $p+2$ doesn't have to be prime) and $p \equiv 5 \pmod{24}$.

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2 Preliminaries

Theorem 2.1. *[3] (Mihăilescu's Theorem) The Diophantine equation* $a^x - b^y = 1$ *has the unique solution* $(a, b, x, y) = (3, 2, 2, 3)$*, where* $a, b, x, y \in \mathbb{N}$ *and* $min\{a, b, x, y\} > 1$.

Theorem 2.2. *[2] The number* 2 *is a quadratic residue of primes of the form* $p = 8k+1$ *and* $p = 8k+7$*. The number* 2 *is not a quadratic residue of primes of the form* $p = 8k + 3$ *and* $p = 8k + 5$.

Lemma 2.3. *Let* p *be prime. Then the Diophantine equation* $1 - p^y = z^2$ *has the unique non-negative integer solution* $(y, z) = (0, 0)$.

Proof. Let $y, z \in \mathbb{N}_0$ such that $1 - p^y = z^2$. Since $z^2 \geq 0$, we have $1 - p^y \geq 0$. Then $y = 0$ from which it follows that $z = 0$. □

Lemma 2.4. *Let* p *be prime. Then the Diophantine equation* $(p+2)^x - 1 = z^2$ *has all non-negative integer solutions* $(p, x, y) \in \{(p, 0, 0), (3, 1, 2)\}$.

Proof. Let $x, z \in \mathbb{N}_0$ such that $(p + 2)^x - 1 = z^2$. If $x = 0$, then $z = 0$ and so $(p, x, z) = (p, 0, 0)$. If $x = 1$, then $p = (z - 1)(z + 1)$. Thus $z - 1 = 1$ and $z + 1 = p$. Then $z = 2$ and $p = 3$. That is, $(p, x, z) = (3, 1, 2)$. For the case $x > 1$, it is easy to check that $z > 1$ which is impossible by Theorem 2.1. □

3 Main results

Theorem 3.1. *Let* p *be prime and* $p \equiv 1 \pmod 4$*. If the Diophantine equation* $(p + 2)^x - p^y = z^2$ *has a positive integer solution, then* x *is even and* $2(p + 2)^{\frac{x}{2}} = p^y + 1$.

Proof. Let $x, y, z \in \mathbb{N}$ such that $(p+2)^x - p^y = z^2$. Since p and $p+2$ are odd, we get $z^2 \equiv 0 \pmod 4$. Thus $0 \equiv (p+2)^x - p^y \equiv [(-1)^x - 1] \pmod 4$. Then $x = 2m$, for some $m \in \mathbb{N}$. Then $p^y = ((p+2)^m - z)((p+2)^m + z)$. Since p is prime, we have $(p+2)^m - z = p^u$ and $(p+2)^m + z = p^{y-u}$, for some $u \in \mathbb{N}_0$. Then $y \geq 2u$ and $2(p+2)^m = p^u(p^{y-2u} + 1)$. If $u > 0$, then $p \mid 2(p+2)^m$ and so $p = 2$ which is impossible since $p \equiv 1 \pmod 4$. Consequently, $u = 0$ and $2(p + 2)^m = p^y + 1$. □

Corollary 3.2. *The Diophantine equation* $15^x - 13^y = z^2$ *has the unique non-negative integer solution* $(x, y, z) = (0, 0, 0)$.

Proof. Let $x, y, z \in \mathbb{N}_0$ such that $15^x - 13^y = z^2$. If $x = 0$ or $y = 0$, then $(x, y, z) = (0, 0, 0)$ by Lemmas 2.3 and 2.4, respectively. For the case $x \geq 1$ and $y \geq 1$, we have x is even and $2 \cdot 15^{\frac{x}{2}} = 13^y + 1$, by Theorem 3.1. This is impossible since $2 \cdot 15^{\frac{x}{2}} \equiv 0 \pmod 3$ and $13^y + 1 \equiv 2 \pmod 3$. □

Corollary 3.3. *The Diophantine equation* $19^x - 17^y = z^2$ *has the unique non-negative integer solution* $(x, y, z) = (0, 0, 0)$.

Proof. Let $x, y, z \in \mathbb{N}_0$ such that $19^x - 17^y = z^2$. If $x = 0$ or $y = 0$, then $(x, y, z) = (0, 0, 0)$ by Lemmas 2.3 and 2.4, respectively. For the case $x \geq 1$ and $y \geq 1$, we have $x = 2m$, for some $m \in \mathbb{N}$, and $2 \cdot 19^m = 17^y + 1$, by Theorem 3.1. Since $17^y + 1 \equiv 2 \pmod 8$, we get $2 \cdot 19^m \equiv 2 \pmod 8$. Thus $19^m \equiv 1 \pmod 4$ and so $(-1)^m \equiv 1 \pmod 4$. Then $m = 2n$, for some $n \in \mathbb{N}$. Thus $2 \cdot 2^{2n} \equiv 2 \cdot 19^{2n} \equiv 17^y + 1 \equiv 1 \pmod{17}$. This is impossible since $2 \cdot 2^{2n} \equiv \pm 2, \pm 8 \pmod{17}$. □

Theorem 3.4. *Let p be prime and $p \equiv 5 \pmod{24}$. Then the Diophantine equation $(p+2)^x - p^y = z^2$ has the unique non-negative integer solution $(x, y, z) = (0, 0, 0)$.*

Proof. Let $x, y, z \in \mathbb{N}_0$ such that $(p+2)^x - p^y = z^2$. If $x = 0$ or $y = 0$, then $(x, y, z) = (0, 0, 0)$ by Lemmas 2.3 and 2.4, respectively. Next we consider case $x \geq 1$ and $y \geq 1$. Since $p \equiv 1 \pmod 4$, we have $x = 2m$, for some $m \in \mathbb{N}$, and $2(p+2)^m = p^y + 1$ by Theorem 3.1. Since $p \equiv -1 \pmod 3$, we get $2 \equiv (-1)^y + 1 \pmod 3$. Then $y = 2k$, for some $k \in \mathbb{N}$. Since $p \equiv 5 \pmod 8$, we obtain $2(p+2)^m \equiv 2(-1)^m \pmod 8$ and $p^y + 1 \equiv 5^{2k} + 1 \equiv 2 \pmod 8$. Then $2(-1)^m \equiv 2 \pmod 8$. Then $m = 2t$, for some $t \in \mathbb{N}$ and so $2(p+2)^m = 2(p+2)^{2t} \equiv 2^{2t+1} \pmod p$. Since $p^y + 1 \equiv 1 \pmod p$, we get $2^{2t+1} \equiv 1 \pmod p$ and so $(2^{t+1})^2 \equiv 2 \pmod p$. Hence 2 is a quadratic residue of p, a contradiction to Theorem 2.2 since $p \equiv 5 \pmod 8$. □

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