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A Note on Relatively Commuting Mappings of Prime and Semiprime Rings

Auday Hekmat Mahmood, Maysaa Zaki Salman

Department of Mathematics
College of Education
University of Mustansiriyah
Baghdad, Iraq

email: audaymath@uomustansiriyah.edu.iq,
Maysaa-zaki@uomustansiriyah.edu.iq

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Abstract

The main aim of this work is to present some special results related to relatively commuting mappings. We prove that if $\mathfrak{F}$ is a non trivial derivation on a 2-torsion free prime ring $\mathfrak{R}$, then for any $o \in \mathfrak{R}$, the property that $[[\mathfrak{F}(o), o], \mathfrak{F}(o)]$ is a central force $\mathfrak{F}$ to be commuting on $\mathfrak{R}$.

1 Introduction

Henceforth $\mathfrak{R}$ is an associative ring of characteristic different from 2 and $Z(\mathfrak{R})$ denotes the center of $\mathscr{R}$. A ring $\mathfrak{R}$ with the property that whenever $nu = 0$ with $u \in \mathfrak{R}$ implies that $u = 0$ is called n-torsion free where n is a non-zero integer [1]. Note that the statement "$\mathfrak{R}$ is n-torsion free ring" is equivalent to "$\mathfrak{R}$ is a ring of characteristic different from n". In [2] the concept of relatively commuting mapping was presented as follows:
An additive mapping ρ: $\mathfrak{R} \longmapsto \mathfrak{R}$ is said to be a relatively commuting mapping if for some integer $n \geq 2$ we have $[\rho(u), u]^n = 0$, for all $u \in \mathfrak{R}$. As a special case of relatively commuting mappings when $(n = 1)$, ρ is

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said to be commuting on $\mathscr{R}$[3]. Recall that a ring $\mathfrak{R}$ is said to be have the primeness property if for any u, $y \in \mathfrak{R}$ with $u\mathfrak{R}r = 0$, then either $(= 0)$ or $(r = 0)$ [4]. Equivalently, a prime ring is one where the right annihilator of nonzero ideal is merely 0 while a ring $\mathfrak{R}$ is said to be have the semi primeness property in case $u\mathfrak{R}u = (0)$ leads to $u = 0$ [5]. The mathematical symbol $[u, r]$, for u, $r \in \mathfrak{R}$ denote the usual commutator ur–ru. We frequently use the commutator identities $[u\omega, r] = [u, r]\omega + u[\omega, r]$, and $[u, \omega r] = [u, r] + r[u, \omega]$[6]. By a derivation on $\mathfrak{R}$, we mean a mapping $\mathfrak{F}$: $\mathfrak{R} \longmapsto \mathfrak{R}$ satisfying $\mathfrak{F}(ur) = \mathfrak{F}(u)r + u\mathfrak{F}(r)$ holds for pairs $u, r \in \mathfrak{R}$. Posner [7] gave first important result concerning the centralizing and commuting mappings and showed the existence of non trivial centralizing derivation on a prime ring $\mathfrak{R}$, forces $\mathfrak{R}$ to be commutative. In [8], Posner's Theorem was extended in a different direction that in case $\mathfrak{R}$ is a noncommutative prime ring, any non trivial derivation cannot be centralizing on various subsets of $\mathfrak{R}$. In this paper, we prove some results related to the notion of relatively commuting mapping in prime and semiprime rings.

2 Preliminary results

We shall use the following Jacobi identities on the commutators:

$$[[u, r], w] = [u, [r, w]] - [r, [u, w]]$$
$$[u, [r, w]] = [r, [u, w]] + [w, [r.u]]$$

Furthermore, we shall frequently use the following lemmas:

Lemma 2.1. *[2] Every relatively commuting mapping on a semiprime ring $\mathfrak{R}$ is commuting.*

Lemma 2.2. *[9] Let $\mathfrak{R}$ be a 2-torsion free semiprime ring and let $a, b \in \mathfrak{R}$. Then the following conditions are equivalent:*

- $a\,ub = 0, \forall u \in \mathfrak{R}$
- $b\,ua = 0, \forall u \in \mathfrak{R}$
- $a\,ub + b\,ua = 0, \forall u \in \mathfrak{R}$

Furthermore, $ab = ba = 0$ whenever one of these conditions is fulfilled.

3 Results on Relatively Commuting Mappings

Now, we start with the theorem:

Theorem 3.1. *Let $\mathfrak{R}$ be a semiprime ring of characteristic different from 2 and $\mathfrak{F}$: $\mathfrak{R} \longmapsto \mathfrak{R}$ be a nonzero derivation which satisfies $[[\mathfrak{F}(\mathscr{o}), \mathscr{o}], \mathfrak{F}(\mathscr{o})] = 0$ whenever $\mathscr{o} \in \mathfrak{R}$. Then $\mathfrak{F}$ is a commuting mapping of $\mathfrak{R}$.*

Proof.
In view of our hypothesis:

$$[[\mathfrak{F}(\mathscr{o}), \mathscr{o}], \mathfrak{F}(\mathscr{o})] = 0, \forall \mathscr{o} \in \mathfrak{R}. \tag{1}$$

Now, define $\mathfrak{Y}(\mathscr{o}) = [\mathfrak{F}(\mathscr{o}), \mathscr{o}]$. Then the above relation becomes:

$$[\mathfrak{Y}(\mathscr{o}), \mathfrak{F}(\mathscr{o})] = 0, \forall \mathscr{o} \in \mathfrak{R}. \tag{2}$$

Polarizing the relation (1), we get:

$$\begin{aligned}&[\mathfrak{Y}(\mathscr{o}), \mathfrak{F}(\mathscr{r})] + [\mathfrak{Y}(\mathscr{r}), \mathfrak{F}(\mathscr{o})] + [[\mathfrak{F}(\mathscr{o}), \mathscr{r}], \mathfrak{F}(\mathscr{o})] + [[\mathfrak{F}(\mathscr{o}), \mathscr{r}], \mathfrak{F}(\mathscr{r})] \\ &\quad + [[\mathfrak{F}(\mathscr{r}), \mathscr{o}], \mathfrak{F}(\mathscr{o})] + [[\mathfrak{F}(\mathscr{r}), \mathscr{o}], \mathfrak{F}(\mathscr{r})] = 0, \forall \mathscr{o}, \mathscr{r} \in \mathfrak{R}.\end{aligned}$$

Replacing $\mathscr{o}$ by $2\mathscr{o}$, comparing the obtained relation with the last one, and after using the 2-torsion free $\mathfrak{R}$, we obtain:

$$[\mathfrak{Y}(\mathscr{o}), \mathfrak{F}(\mathscr{r})] + [[\mathfrak{F}(\mathscr{o}), \mathscr{r}], \mathfrak{F}(\mathscr{o})] + [[\mathfrak{F}(\mathscr{r}), \mathscr{o}], \mathfrak{F}(\mathscr{o})] = 0, \forall \mathscr{o}, \mathscr{r} \in \mathfrak{R}. \tag{3}$$

Taking $\mathscr{o}\mathscr{r}$ instead of $\mathscr{r}$ in (3) gives:

$$\begin{aligned}&[\mathfrak{Y}(\mathscr{o}), \mathfrak{F}(\mathscr{o})]\mathscr{r} + \mathfrak{F}(\mathscr{o})[\mathfrak{Y}(\mathscr{o}), \mathscr{r}] + [\mathfrak{Y}(\mathscr{o}), \mathscr{o}]\mathfrak{F}(\mathscr{r}) + \mathscr{o}[\mathfrak{Y}(\mathscr{o}), \mathfrak{F}(\mathscr{r})] + [[\mathfrak{F}(\mathscr{o}), \mathscr{o}], \mathfrak{F}(\mathscr{o})]\mathscr{r} \\ &+ [\mathfrak{F}(\mathscr{o}), \mathscr{o}][\mathscr{r}, \mathfrak{F}(\mathscr{o})] + [\mathscr{o}, \mathfrak{F}(\mathscr{o})][\mathfrak{F}(\mathscr{o}), \mathscr{r}] + \mathscr{o}[[\mathfrak{F}(\mathscr{o}), \mathscr{r}], \mathfrak{F}(\mathscr{o})] + [[\mathfrak{F}(\mathscr{o}), \mathscr{o}], \mathfrak{F}(\mathscr{o})]\mathscr{r} \\ &+ [\mathfrak{F}(\mathscr{o}), \mathscr{o}][\mathscr{r}, \mathfrak{F}(\mathscr{o})] + \mathfrak{F}(\mathscr{o})[[\mathscr{r}, \mathscr{o}], \mathfrak{F}(\mathscr{o})] + [\mathscr{o}, \mathfrak{F}(\mathscr{o})][\mathfrak{F}(\mathscr{r}), \mathscr{o}] + \mathscr{o}[[\mathfrak{F}(\mathscr{r}), \mathscr{o}], \mathfrak{F}(\mathscr{o})] = 0.\end{aligned}$$

According to the relations (1) and (2), we have:

$$\mathfrak{F}(\mathscr{o})[\mathfrak{Y}(\mathscr{o}), \mathscr{r}] + [\mathfrak{Y}(\mathscr{o}), \mathscr{o}]\mathfrak{F}(\mathscr{r}) + \mathfrak{F}(\mathscr{o})[[\mathscr{r}, \mathscr{o}], \mathfrak{F}(\mathscr{o})] + \mathfrak{Y}(\mathscr{o})[\mathscr{o}, \mathfrak{F}(\mathscr{r})] + 3\mathfrak{Y}(\mathscr{o})[\mathscr{r}, \mathfrak{Y}(\mathscr{o})] = 0. \tag{4}$$

Let us take $\mathscr{r}\mathscr{o}$ instead of $\mathscr{r}$ in (4). Then:

$$\mathfrak{F}(x)y[\mathfrak{Y}(x),x]+\mathfrak{F}(x)[\mathfrak{Y}(x),y]x+[\mathfrak{Y}(x),x]\mathfrak{F}(y)x+[\mathfrak{Y}(x),x]y\mathfrak{F}(x)$$
$$+\mathfrak{F}(x)[[y,x],\mathfrak{F}(x)]x+\mathfrak{F}(x)[y,x][u,\mathfrak{F}(x)]+\mathfrak{Y}(x)[x,\mathfrak{F}(y)]x$$
$$+\mathfrak{Y}(x)[x,\mathfrak{F}(y)]+\mathfrak{Y}(x)y[x,\mathfrak{F}(x)]+\mathfrak{Y}(x)[x,y]\mathfrak{F}(x)+3\mathfrak{Y}(x)[y,\mathfrak{F}(x)]x$$
$$+3\mathfrak{Y}(x)y[x,\mathfrak{F}(x)]=0,\forall x,y\in\mathfrak{R}.$$

Comparing the new relation with (4) implies that:

$$\begin{aligned}&\mathfrak{F}(x)y[\mathfrak{Y}(x),x]+[\mathfrak{Y}(x),x]y\mathfrak{F}(x)+\mathfrak{Y}(x)[x,y]\mathfrak{F}(x)\\&-\mathfrak{F}(x)[y,x]\mathfrak{Y}(x)-4\mathfrak{Y}(x)yf(x)=0.\end{aligned}\tag{5}$$

Putting $y\mathfrak{F}(x)$ instead of y in (5), we find:

$$\mathfrak{F}(x)y\mathfrak{F}(x)[\mathfrak{Y}(x),x]+[\mathfrak{Y}(x),x]y\mathfrak{F}(x)^2+\mathfrak{Y}(x)[x,y]\mathfrak{F}(x)^2+\mathfrak{F}(x)y[x,\mathfrak{F}(x)]\mathfrak{F}(x)$$
$$-\mathfrak{F}(x)y[\mathfrak{F}(x),x]\mathfrak{Y}(x)-\mathfrak{F}(x)[y,x]\mathfrak{F}(x)\mathfrak{Y}(x)-4\mathfrak{Y}(x)y\mathfrak{F}(x)\mathfrak{Y}(x)=0,\forall x,y\in\mathfrak{R}.$$

In view of relations (1) and (5), the above statement becomes:

$$\begin{aligned}&\mathfrak{F}(x)y\mathfrak{F}(x)[\mathfrak{Y}(x),x]-\mathfrak{Y}(x)y\mathfrak{Y}(x)\mathfrak{F}(x)-\mathfrak{F}(x)y\mathfrak{Y}(x)^2\\&-\mathfrak{F}(x)y[\mathfrak{Y}(x),x]\mathfrak{F}(x)=0,\forall x,y\in\mathfrak{R}.\end{aligned}$$

That is,

$$\begin{aligned}&\mathfrak{Y}(x)y\mathfrak{Y}(x)\ \mathfrak{F}(x)\ +\mathfrak{F}(x)\ y\ [[\mathfrak{Y}(x),\ x],\mathfrak{F}(x)]\\&+\mathfrak{F}(x)y\mathfrak{Y}(x)^2=\ 0,\ \forall\ x\ ,\ y\ \in\ \mathfrak{R}.\end{aligned}$$

Making use of Jacobi identity for commutators together with relation (2), we see that the second term in the above relation is equal to zero. So the above relation becomes:

$$\mathfrak{Y}(x)\,y\,\mathfrak{Y}(x)\,\mathfrak{F}(x)+\ \mathfrak{F}(x)\,y\,\mathfrak{Y}(x)^2\ =\ 0,\ \forall\ x\ ,\ y\ \in\ \mathfrak{R}.\tag{6}$$

Pre multiplication of (6) by $\mathfrak{Y}(x)$ leads to:
$\mathfrak{Y}(x)^2\ y\ \mathfrak{Y}(x)\ \mathfrak{F}(x)\ +\ \mathfrak{Y}(x)\ \mathfrak{F}(x)\ y\ \mathfrak{Y}(x)^2\ =\ 0,\ \ \forall\ x\ ,\ y\ \in\ \mathfrak{R}.$
Now:

$$\mathfrak{Y}(x)^2y\mathfrak{Y}(x)\mathfrak{F}(x)=0,\forall x,\ y\in\mathfrak{R}.\tag{7}$$

According to the relation (2), we can write $\mathfrak{F}(o)\mathfrak{Y}(o)$ instead of $\mathfrak{Y}(o)\mathfrak{F}(o)$. So:

$$\mathfrak{Y}(o)^2 r\mathfrak{F}(o)\mathfrak{Y}(o) = 0, \forall o, r \in \mathfrak{R}. \tag{8}$$

Now, in the relation (8), the substitution ro for r once and post multiplication by o in another implies that:

$$\mathfrak{Y}(o)^2 r\ o\ \mathfrak{F}(o)\mathfrak{Y}(o) \ = \ 0, \ \forall o, \ r \ \in \ \mathfrak{R}. \tag{9}$$

$$\mathfrak{Y}(o)^2 r\mathfrak{F}(o)\mathfrak{Y}(o)o = 0, \forall o, r \in \mathfrak{R}. \tag{10}$$

Subtracting the relation (9) from (10), we get:

$$\mathfrak{Y}(o)^2 r[\mathfrak{F}(o)\mathfrak{Y}(o), o] = 0, \forall o, r \in \mathfrak{R}.$$

$$\mathfrak{Y}(o)^2 r\mathfrak{F}(o)[\mathfrak{Y}(o), o] + \mathfrak{Y}(o)^2 r\mathfrak{Y}(o)^2 = 0, \forall o, r \in \mathfrak{R}. \tag{11}$$

Putting $r\mathfrak{Y}(o)$ instead of r in (11), in view of (7), we obtain:

$$\mathfrak{Y}(o)^2 r\mathfrak{Y}(o)^3 = 0, \forall o, r \in \mathfrak{R}. \tag{12}$$

Multiplying Equation (12) from the left by $\mathfrak{Y}(o)$ yields:

$$\mathfrak{Y}(o)^3 r\mathfrak{Y}(o)^3 = 0, \forall o, r \in \mathfrak{R}.$$

Using the semi primeness property of $\mathfrak{R}$, we get:

$$\mathfrak{Y}(o)^3 = 0, \forall o \ \in \mathfrak{R}.$$

That is,

$$[\mathfrak{F}(o), o]^3 = 0, \forall o \in \mathfrak{R}.$$

Hence $\mathfrak{F}$ is a relatively commuting mapping on $\mathfrak{R}$. Applying lemma 2.1, it follows that $\mathfrak{F}$ is a commuting mapping.

Theorem 3.2. *Let $\mathfrak{R}$ be a semiprime ring of characteristic different from 2, $\mathfrak{F}$: $\mathfrak{R} \longmapsto \mathfrak{R}$ a nonzero derivation which satisfies $[[\mathfrak{F}(o), o], \mathfrak{F}(o)]$ lies in the central of $\mathfrak{R}$ whenever $o \in \mathfrak{R}$. Then $\mathfrak{F}$ is a commuting map on $\mathfrak{R}$.*

Proof.

According to our hypothesis:

$$[[\mathfrak{F}(o), o], \mathfrak{F}(o)] \in \ Z(\mathfrak{R}), \forall o \in \mathfrak{R}. \tag{13}$$

Taking $\mathscr{o}+\mathscr{r}$ instead of $\mathscr{u}$ in (13), we get:

$$[[\mathfrak{F}(\mathscr{o}),\mathscr{o}],\mathfrak{F}(\mathscr{r})]+[[\mathfrak{F}(\mathscr{o}),\mathscr{r}],\mathfrak{F}(\mathscr{o})]+[[\mathfrak{F}(\mathscr{o}),\mathscr{r}],\mathfrak{F}(\mathscr{r})]+[[\mathfrak{F}(\mathscr{r}),\mathscr{o}],\mathfrak{F}(\mathscr{o})]$$
$$+[[\mathfrak{F}(\mathscr{r}),\mathscr{o}],\mathfrak{F}(\mathscr{r})]+[[\mathfrak{F}(\mathscr{r}),\mathscr{r}],\mathfrak{F}(\mathscr{o})] \in Z(\mathscr{R}),\forall \mathscr{o},\mathscr{r}\in\mathfrak{R}. \tag{14}$$

The substitution $2\mathscr{r}$ for $\mathscr{r}$ and comparing the new relation with (14), we see:

$$[[\mathfrak{F}(\mathscr{o}),\mathscr{o}],\mathfrak{F}(\mathscr{r})]+[[\mathfrak{F}(\mathscr{o}),\mathscr{r}],(\mathscr{o})]+[[\mathfrak{F}(\mathscr{r}),\mathscr{o}],\mathfrak{F}(\mathscr{o})] \in \mathrm{Z}(\mathscr{R}),\forall\ \mathscr{o},\mathscr{r}\in\mathfrak{R}.$$

Putting $\mathscr{r}=\mathscr{o}^2$ in the last statement gives:

$$[[\mathfrak{F}(\mathscr{o}),\mathscr{o}],\mathfrak{F}(\mathscr{o})]\mathscr{o}+\mathfrak{F}(\mathscr{o})[[\mathfrak{F}(\mathscr{o}),\mathscr{o}],\mathscr{o}]+\mathscr{o}[[\mathfrak{F}(\mathscr{o}),\mathscr{o}],\mathfrak{F}(\mathscr{o})]+[[\mathfrak{F}(\mathscr{o}),\mathscr{o}],\mathscr{o}]\mathfrak{F}(\mathscr{o})$$
$$+\mathscr{o}[[\mathfrak{F}(\mathscr{o}),\mathscr{o}],\mathfrak{F}(\mathscr{o})]+[\mathfrak{F}(\mathscr{o}),\mathscr{o}][\mathscr{o},\mathfrak{F}(\mathscr{o})]+[[\mathfrak{F}(\mathscr{o}),\mathscr{o}],\mathfrak{F}(\mathscr{o})]\mathscr{o}+[\mathfrak{F}(\mathscr{o}),\mathscr{o}][\mathscr{o},\mathfrak{F}(\mathscr{o})]$$
$$+[[\mathfrak{F}(\mathscr{o}),\mathscr{o}],\mathfrak{F}(\mathscr{o})]\mathscr{o}+[\mathfrak{F}(\mathscr{o}),\mathscr{o}][\mathscr{o},\mathfrak{F}(\mathscr{o})]+\mathscr{o}[[\mathfrak{F}(\mathscr{o}),\mathscr{o}],\mathfrak{F}(\mathscr{o})]$$
$$+[\mathscr{o},\mathfrak{F}(\mathscr{o})][\mathfrak{F}(\mathscr{o}),\mathscr{o}] \in \mathrm{Z}(\mathfrak{R}),\forall\ \mathscr{o}\in\mathfrak{R}.$$

Consequently,

$$6[[[\mathfrak{F}(\mathscr{o}),\mathscr{o}],\mathfrak{F}(\mathscr{o})]\mathscr{o},\mathfrak{F}(\mathscr{o})]-4[[\mathfrak{F}(\mathscr{o}),\mathscr{o}]2,\mathfrak{F}(\mathscr{o})]+[\mathfrak{F}(\mathscr{o})[[\mathfrak{F}(\mathscr{o}),\mathscr{o}],\mathscr{o}],\mathfrak{F}(\mathscr{o})]$$
$$+[[[\mathfrak{F}(\mathscr{o}),\mathscr{o}],\mathscr{o}]\mathfrak{F}(\mathscr{o}),\mathfrak{F}(\mathscr{o})]=0,\forall\ \mathscr{o}\in\mathfrak{R}.$$

In view of (13) and the commutators identities, the last relation can be written as:

$$14[[\mathfrak{F}(\mathscr{o}),\mathscr{o}],\mathfrak{F}(\mathscr{o})][\mathfrak{F}(\mathscr{o}),\mathscr{o}]+\mathfrak{F}(\mathscr{o})[[[\mathfrak{F}(\mathscr{o}),\mathscr{o}],\mathscr{o}],\mathfrak{F}(\mathscr{o})]+[[[\mathfrak{F}(\mathscr{o}),\mathscr{o}],\mathscr{o}],\mathfrak{F}(\mathscr{o})]\mathfrak{F}(\mathscr{o})=0. \tag{15}$$

An application of Jacobian identity for commutators yields:

$$14[[(\mathscr{o}),\mathscr{o}],\mathfrak{F}(\mathscr{o})][\mathfrak{F}(\mathscr{o}),\mathscr{o}]=0,\forall\ \mathscr{o}\in\mathfrak{R}.$$

Since $\mathscr{R}$ is 2-torsion free, we have:

$$[[\mathfrak{F}(\mathscr{o}),\mathscr{o}],\mathfrak{F}(\mathscr{o})][\mathfrak{F}(\mathscr{o}),\mathscr{o}]=0,\forall\ \mathscr{o}\in\mathfrak{R}. \tag{16}$$

By (13), left multiplication of (16) by $\mathscr{z}$ gives:

$$[[\mathfrak{F}(\mathscr{o}),\mathscr{o}],\mathfrak{F}(\mathscr{o})]\mathscr{z}[(\mathscr{o}),\mathscr{o}]=0,\forall\mathscr{o}\in\mathfrak{R}.$$

Since $\mathfrak{R}$ is prime, we have either $[\mathfrak{F}(\mathscr{o}),\mathscr{o}]=0,\forall\mathscr{o}\in\mathfrak{R}($ that is, $\mathfrak{F}$ is a commuting mapping) or

$$[\mathfrak{F}(\mathscr{o}),\mathscr{o}],\mathfrak{F}(\mathscr{o})]=0,\forall\ \mathscr{o}\in\mathfrak{R}.$$

Again, by using Theorem (3.1), we conclude that $\mathfrak{F}$ is a commuting mapping on $\mathfrak{R}$.

Theorem 3.3. *Let $\mathfrak{R}$ be a prime ring of characteristic different from 2 and $\mathfrak{P}$, $\mathfrak{F}$: $\mathfrak{R} \longmapsto \mathfrak{R}$ are derivations which satisfy $[[\mathfrak{P}(o), o], \mathfrak{F}(o)] = 0$, for all $o \in \mathfrak{R}$. If $\mathfrak{F}$ is centralizing on $\mathfrak{R}$, then either $\mathfrak{P}$ or $\mathfrak{F}$ is commuting on $\mathfrak{R}$.*

Proof.
Choose any $o \in \mathfrak{R}$. Then:

$$[[\mathfrak{P}(o), o], \mathfrak{F}(o)] = 0 \tag{17}$$

Now, let $\mathfrak{A} : \mathscr{R} \longrightarrow \mathscr{R}$ be a mapping defined by $\mathfrak{A}(o) = [\mathfrak{P}(o), o]$. It follows that:

$$[\mathfrak{A}(o), \mathfrak{F}(o)] = 0, \forall o \in \mathfrak{R}. \tag{18}$$

Replacing o with $o + r$ in (18), we find that:

$$\begin{aligned}[\mathfrak{A}(o), \mathfrak{F}(r)] + [\mathfrak{A}(r), \mathfrak{F}(o)] + [[\mathfrak{P}(o), r], \mathfrak{F}(o)] + [[\mathfrak{P}(o), r], \mathfrak{F}(r)]\\ +[[\mathfrak{P}(r), o], \mathfrak{F}(o)] + [[\mathfrak{P}(r), o], \mathfrak{F}(r)] = 0, \forall o, r \in \mathfrak{R}.\end{aligned} \tag{19}$$

Substituting $-o$ for o in (19), then comparing the obtained relation with (19), we get:

$$[\mathfrak{A}(o), \mathfrak{F}(r)] + [[\mathfrak{P}(o), r], \mathfrak{F}(o)] + [[\mathfrak{P}(r), o], \mathfrak{F}(o)] = 0, \forall o, r \in \mathfrak{R}. \tag{20}$$

Substituting rz instead of r in (20) leads to:

$$\begin{aligned}&[\mathfrak{A}(o), \mathfrak{F}(r)]z + \mathfrak{F}(r)[\mathfrak{A}(o), z] + r[\mathfrak{A}(o), \mathfrak{F}(z)] + [\mathfrak{A}(o), r]\mathfrak{F}(z) + [[\mathfrak{P}(o), r], \mathfrak{F}(o)]z\\ &+ [\mathfrak{P}(r), o[z, \mathfrak{F}(o)] + r[[\mathfrak{P}(o), z], \mathfrak{F}(o)] + [r, \mathfrak{F}(o)][\mathfrak{P}(o), z] + [[\mathfrak{P}(r), o], \mathfrak{F}(o)]z\\ &+ [\mathfrak{P}(r), o][z, \mathfrak{F}(o)] + \mathfrak{P}(r)[[z, o], \mathfrak{F}(o)] + [\mathfrak{P}(r), \mathfrak{F}(o)][z, o] + [[r, o], \mathfrak{F}(o)]f(z)\\ &+ [r, o][\mathfrak{P}(z), \mathfrak{F}(o) - +r[[\mathfrak{P}(z), o], \mathfrak{F}(o)] + [r, \mathfrak{F}(o)][\mathfrak{P}(z), o] = 0, \forall o, r, z \in \mathfrak{R}.\end{aligned}$$

According to (20), the last statement can be given by:

$$\begin{aligned}&\mathfrak{F}(r)\ [\mathfrak{A}(o), z] + [\mathfrak{A}(o), r]\mathfrak{F}(z) + [\mathfrak{P}(r), o][z, \mathfrak{F}(o)] + [r, \mathfrak{F}(o)][\mathfrak{P}(o), z]\\ &+ [\mathfrak{P}(r), o][z, \mathfrak{F}(o)] + \mathfrak{P}(r)[[z, o], \mathfrak{F}(o)] + [\mathfrak{P}(r), o][z, \mathfrak{F}(o)] + [[r, o], \mathfrak{F}(o)]\mathfrak{P}(z)\\ &+ [r, o][\mathfrak{P}(z), \mathfrak{F}(o)] + [r, \mathfrak{F}(o)][\mathfrak{P}(z), o] = 0, \forall o, r, z \in \mathfrak{R}.\end{aligned}$$

Taking $z = o$ in the last relation, we have:

$$\begin{aligned}&\mathfrak{F}(r)[\mathfrak{A}(o), o] + [\mathfrak{P}(o), r][o, \mathfrak{F}(o)] + [\mathfrak{A}(o), r]\mathfrak{F}(o) + [r, \mathfrak{F}(o)][\mathfrak{P}(o), o]\\ &+ [\mathfrak{P}(r), o][o, \mathfrak{F}(o)] + [[r, o], \mathfrak{F}(o)]\mathfrak{P}(o) + [r, o][\mathfrak{P}(o), \mathfrak{F}(o)] + [r, \mathfrak{F}(o)][\mathfrak{P}(o), o]\\ &= 0, \forall o, r \in \mathfrak{R}.\end{aligned} \tag{21}$$

Again, putting $r\omega$ instead of r in (21), we get:

$$\mathfrak{F}(r)\omega[\mathfrak{A}(o),o]+r\mathfrak{F}(\omega)[\mathfrak{A}(o),o]+r[\mathfrak{P}(o),\omega][o,\mathfrak{F}(o)]+[\ \mathfrak{P}(o),r]\omega[o,\mathfrak{F}(o)]$$
$$+r[\mathfrak{A}(o),\omega]\mathfrak{F}(o)+[\mathfrak{A}(o),r]\omega\mathfrak{F}(o)+r[\omega,\mathfrak{F}(o)][\mathfrak{P}(o),o]+[r,\mathfrak{F}(o)]\omega[\mathfrak{P}(o),o]$$
$$+\mathfrak{P}(r)[\omega,o][o,\mathfrak{F}(o)]+[\ \mathfrak{P}(r),o]\omega[o,\mathfrak{F}(o)]+r[\mathfrak{P}(\omega),o][o,\mathfrak{F}(o)]+[r,o]\mathfrak{P}(\omega)[o,\mathfrak{F}(o)]$$
$$+[[r,o],\mathfrak{F}(o)]\omega\mathfrak{P}(o)+[r,o][\omega,\mathfrak{F}(o)]\mathfrak{P}(o)+r[[\omega,o],\mathfrak{F}(o)]\mathfrak{P}(u)$$
$$+[r,\mathfrak{F}(o)][\omega,o]\mathfrak{P}(o)+r[\omega,\mathfrak{F}(o)][\mathfrak{P}(o),o]+[r,\mathfrak{F}(o)]\omega[\mathfrak{P}(o),o]+$$
$$r[\omega,o][\mathfrak{P}(o),\mathfrak{F}(o)]+[r,o]\omega[\mathfrak{P}(o),\mathfrak{F}(o)]=0,\forall o,r,\omega\in\mathfrak{R}.$$

In view of (21) the obtained relation becomes:

$$\mathfrak{F}(r)\omega[\mathfrak{A}(o),o]+[\mathfrak{A}(o),r]\omega\mathfrak{F}(o)+[\mathfrak{P}(o),r]\omega[o,\mathfrak{F}(o)]+[r,\mathfrak{F}(o)]\omega[\mathfrak{P}(o),o]$$
$$+\mathfrak{P}(r)[\omega,o][o,\mathfrak{F}(o)]+[\mathfrak{P}(r),o]\omega[o,\mathfrak{F}(o)]+\ [r,\ o]\mathfrak{P}(\omega)[o,\mathfrak{F}(o)]+[[r,o],\mathfrak{F}(o)]\omega\mathfrak{P}(o)$$
$$+[r,\mathfrak{F}(o)][\omega,o]\mathfrak{P}(o)+[r,\mathfrak{F}(o)]\omega[\mathfrak{P}(o),o]+[r,o]\omega[\mathfrak{P}(o),\mathfrak{F}(o)]=0,\forall\ o,r,\omega\in\mathfrak{R}.$$

Substituting o instead of r gives:

$$\mathfrak{F}(o)\omega[\mathfrak{A}(o),o]+[\mathfrak{A}(o),o]\omega\mathfrak{F}(o)+2[\mathfrak{P}(o),o]\omega[o,\mathfrak{F}(o)]$$
$$+2[o,\mathfrak{F}(o)]\omega[\mathfrak{P}(o),o]+\mathfrak{P}(o)[\omega,o][o,\mathfrak{F}(o)]+[o,\mathfrak{F}(o)][\omega,o]\mathfrak{P}(o)=0,\forall o,\omega\in\mathfrak{R}. \tag{22}$$

Replacing ω by $\omega\mathfrak{F}(o)$ in (22), we obtain:

$$\mathfrak{F}(o)\omega\mathfrak{F}(o)[\mathfrak{A}(o),o]+[\mathfrak{A}(o),o]\omega\mathfrak{F}(o)^2+2\mathfrak{A}(o)\omega\mathfrak{F}(o)[o,\mathfrak{F}(o)]$$
$$+2[o,\mathfrak{F}(o)]\omega\mathfrak{F}(o)\mathfrak{A}(o)-\ \mathfrak{P}(o)\ \omega\ \ [o,\ \mathfrak{F}(o)]^2+\mathfrak{P}(o)[\omega,o]\mathfrak{F}(o)[o,\mathfrak{F}(o)]$$
$$+[o,\mathfrak{F}(o)]\omega[\mathfrak{F}(o),o]\mathfrak{P}(o)+[o,\mathfrak{F}(o)][\omega,o]\mathfrak{F}(o)\mathfrak{P}(o)=0,\forall o,\omega\in\mathfrak{R}.$$

Comparing the new relation with (22), we obtain:

$$\mathfrak{F}(o)\omega[\mathfrak{F}(o),[\mathfrak{A}(o),o]]+2\mathfrak{A}(o)\omega[\mathfrak{F}(o),[o,\mathfrak{F}(o)]]+\mathfrak{P}(o)\omega[o,\mathfrak{F}(o)]^2$$
$$+\ \mathfrak{P}(o)[\omega,o][\mathfrak{F}(o),[o,\mathfrak{F}(o)]]+[o,\mathfrak{F}(o)]\omega[\mathfrak{F}(o),o]\ \mathfrak{P}(o)=0,\forall\ o,\omega\in\mathfrak{R}.$$

Using the Jacobi identity on the commutators and the fact that $\mathfrak{F}$ is centralizing on $\mathscr{R}$, the last relation reduces to:

$$\mathfrak{P}(o)\ \omega\ \ [o,\ \mathfrak{F}\,(o)]^2+[o,\mathfrak{F}(o)]^2\omega\mathfrak{P}(o)=0,\forall o,\omega\in\mathfrak{R}.$$

An application of Lemma 2.2 leads to:

$$\mathfrak{P}(o)\ \omega\ \ [o,\ \mathfrak{F}(o)]^2=0,\forall o,\omega\in\mathfrak{R}. \tag{23}$$

Putting $\sigma\omega$ for ω in (23) leads to:

$$\mathfrak{P}(\sigma)\sigma\omega[\sigma,\mathfrak{F}(\sigma)]^2 = 0, \forall \ \sigma,\omega \in \mathfrak{R}. \tag{24}$$

Post multiplication of (23) by σ, and subtracting the relation so obtained from (8) leads to:

$$[\mathfrak{P}(\sigma),\sigma]\omega[\sigma,\mathfrak{F}(\sigma)]^2 = 0, \forall \sigma,\omega \in \mathfrak{R}.$$

Using the prime property of $\mathfrak{R}$, we conclude that either:

$$[\mathfrak{F}(\sigma)\sigma]^2 = 0, \forall \sigma \in \mathfrak{R}.$$

(That is, $\mathfrak{F}$ is a relatively commuting mapping and, consequently, a commuting mapping on $\mathfrak{R}$ by lemma (2.1)) or

$$[\mathfrak{P}(\sigma),\sigma] = 0, \forall \ \sigma \in \mathfrak{R}.$$

Therefore $\mathfrak{P}$ is a commuting map.

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