International Journal of Mathematics and
Computer Science, **18**(2023), no. 2, 359–367

Geraghty type generalized F-contraction for dislocated quasi-metric spaces

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(Received June 12, 2022, Revised January 4, 2023,
Accepted January 15, 2023, Published January 23, 2023)

Abstract

In this paper, we explore the existence and uniqueness of fixed points for the new constructed contraction mapping on dislocated quasi-metric spaces by using Geraghty contraction and F-contraction. Moreover, we support our results by a couple of non-trivial examples.

1 Introduction

Let Ω be the family of all functions $\beta : [0,\infty) \to [0,1)$ which satisfy the condition

$$\lim_{n\to\infty} \beta(t_n) = 1 \;\text{ implies }\; \lim_{n\to\infty} t_n = 0. \tag{1.1}$$

Key words and phrases: F-contraction, Dislocated quasi-metric space, Geraghty type contraction mapping.
AMS (MOS) Subject Classifications: 47H09, 47H10.
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ISSN 1814-0432, 2023, http://ijmcs.future-in-tech.net

Using such a function, Geraghty [1] proved the following theorem:

Theorem 1.1. [1] *Let (X,d) be a complete metric space and let T be a self-mapping on X. Suppose that there exists $\beta \in \Omega$ such that, for all $u, v \in X$,*

$$d(Tu, Tv) \leq \beta(d(u,v))d(u,v), \tag{1.2}$$

then T has a unique fixed point $z \in X$ and $\{T^n z\}$ converges to z for all $z \in X$.

Many authors have discovered this theorem as can be seen in [6, 7, 8, 9].

Definition 1.2. [2] *Let (X,d) be a metric space. The mapping $T : X \to X$ is called an F-contraction, if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $u, v \in X$,*

$$d(Tu, Tv) > 0 \Rightarrow \tau + F(d(Tu, Tv)) \leq F(d(u,v)), \tag{1.3}$$

where $F : \mathbb{R}^+ \to \mathbb{R}$ is strictly increasing $\lim_{n\to\infty} F(\alpha_n) = -\infty$ if and only if $\lim_{n\to\infty} \alpha_n = 0$ and there exists a number $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = -\infty$.

The family of all functions $F : (0,\infty) \to \mathbb{R}$ is denoted by $\mathcal{F}$ if F satisfies the following conditions:

- $(F1)$ *F is strictly increasing;*
- $(F2)$ *For every sequence $\{\alpha_n\}$ in $(0,\infty)$, we have $\lim_{n\to\infty} F(\alpha_n) = -\infty$ if and only if $\lim_{n\to\infty} \alpha_n = 0$;*
- $(F3)$ *There exists a number $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = -\infty$.*

Definition 1.3. [4] *Let X be a nonempty set and let $d : X \times X \to \mathbb{R}^+$ be a function such that the following are satisfied:*

- (i) *$d(u,v) = d(v,u) = 0$ implies that $u = v$;*
- (ii) *$d(u,v) \leq d(u,w) + d(w,v)$ for all $u, v, w \in X$.*

Then d is called dislocated quasi-metric on X and the pair (X,d) is called a dislocated quasi-metric space.

Definition 1.4. [3] *Let $T : X \to X$ be a self-mapping and let $\alpha : X \times X \to \mathbb{R}^+$ be a function. Then T is said to be triangular α-orbital admissible if T is α-orbital admissible and $\alpha(u,v) \geq 1$, $\alpha(v,Tv) \geq 1$ imply $\alpha(u,Tv) \geq 1$.*

Lemma 1.5. [3] *Let $T : X \to X$ be a triangular α-orbital admissible mapping. Assume that there exists $u_1 \in X$ such that $\alpha(u_1, Tu_1) \geq 1$. Define a sequence $\{u_n\}$ by $u_{n+1} = Tu_n$. Then $\alpha(u_n, u_m) \geq 1$ for all $m, n \in \mathbb{N}$ with $n < m$.*

Definition 1.6. [5] *Let $T : X \to X$ be a self-mapping on a metric space. For each $u \in X$ and for any positive whole number n,*

$$O_T(u, n) = \{u, Tu, \ldots, T^n u\} \quad \text{and} \quad O_T(u, \infty) = \{u, Tu, \ldots, T^n u, \ldots\}.$$

The set $O_T(u, \infty)$ is called the orbit of T at x and the metric space X is called T-orbitally complete if every Cauchy sequence in $O_T(u, \infty)$ is convergent in X.

The purpose of this paper is to prove some fixed point results in dislocated quasi-metric space using a Geraghty type generalized F-contraction.

2 Main results

Definition 2.1. *Let (X, d) be a dislocated quasi-metric space and let $\alpha : X \times X \to \mathbb{R}^+$ be a function. A self-mapping $T : X \to X$ is called an (α, β, F)-Geraghty type contraction mapping if there exists $\beta \in \Omega$ such that, for all $u, v \in X$, with $\tau > 0$, $d(Tu, Tv) > 0$ and $\alpha(u, v) \geq 1$,*

$$\alpha(u, v)(\tau + F(d(Tu, Tv))) \leq \beta(M_T(u, v))F(M_T(u, v)), \tag{2.4}$$

where

$$M_T(u, v) = \max\left\{d(u, Tu), d(v, Tv), \frac{(1 + d(u, Tu))d(v, Tv)}{1 + d(u, v)}\right\}.$$

Theorem 2.2. *Let (X, d) be a T-orbitally complete dislocated quasi-metric space such that $T : X \to X$ is a self-mapping. Suppose $\alpha : X \times X \to \mathbb{R}^+$ is a function satisfying the following conditions:*

(i) *T is an (α, β, F)-Geraghty type contraction mapping;*

(ii) *T is triangular α-orbital admissible mapping;*

(iii) *There exists $u_1 \in X$ such that $\alpha(u_1, Tu_1) \geq 1$.*

Then T has a fixed point $z \in X$ and $\{T^n u_1\}$ converges to z.

Proof. Let $u_1 \in X$ such that $\alpha(u_1, Tu_1) \geq 1$. Define a sequence $\{u_n\}$ by $u_{n+1} = T^n u$, for $n \geq 1$. If $u_n = u_{n+1}$ for some n, then obviously T has a fixed point. Consequently, throughout the proof, we suppose that $u_n \neq u_{n+1}$ for all $n \geq 1$. By Lemma 1.5, used recursively, we have

$$\alpha(u_n, u_{n+1}) \geq 1 \quad \forall_n \geq 1. \tag{2.5}$$

By (2.4), we get

$$\begin{aligned} \tau + F(d(T^n u, T^{n+1}u))) &\leq \tau + F(d(T^{n-1}u, T^n u))) \\ &\leq \alpha(T^{n-1}u, T^n u)(\tau + F(d(TT^{n-1}u, TT^n u))) \\ &\leq \beta(M_T(T^{n-1}u, T^n u))F(M_T(T^{n-1}u, T^n u)), \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} &M_T(T^{n-1}u, T^n u) \\ &= \max\left\{ d(T^{n-1}u, T^n u), d(T^n u, T^{n+1}u), \frac{(1 + d(T^{n-1}u, T^n u))d(T^n u, T^{n+1}u)}{1 + d(T^{n-1}u, T^n u)} \right\} \\ &= \max\{d(T^{n-1}u, T^n u), d(T^n u, T^{n+1}u)\}. \end{aligned}$$

The assertion $M_T(T^{n-1}u, T^n u) = d(T^n u, T^{n+1}u)$ is not true. This is because

$$\tau + F(d(T^n u, T^{n+1}u))) < F(d(T^n u, T^{n+1}u)) \tag{2.7}$$

is a contradiction. Consequently, $d(T^n u, T^{n+1}u) < d(T^{n-1}u, T^n u)$. Thus,

$$\tau + F(d(T^n u, T^{n+1}u))) < F(d(T^{n-1}u, T^n u)) \tag{2.8}$$

or

$$F(d(T^n u, T^{n+1}u))) \leq F(d(T^{n-1}u, T^n u)) - \tau. \tag{2.9}$$

In general, one can get

$$F(d(T^n u, T^{n+1}u))) \leq F(d(T^{n-1}u, T^n u)) - n\tau. \tag{2.10}$$

Letting $n \to \infty$ in (2.10) shows that $\lim_{n\to\infty} F(d(T^{n-1}u, T^n u)) = -\infty$. Hence

$$\lim_{n\to\infty} d(T^{n-1}u, T^n u) = 0. \tag{2.11}$$

Suppose that the sequence $\{u_n\}$ is not Cauchy. Then there exists $\epsilon > 0$ and we can define two subsequences $\{T^{m_l}u\}$ and $\{T^{n_l}u\}$ of the sequence

$\{T^n u\}$ such that, for any $n_l > m_l > l$, $d(T^{m_l}u, T^{n_l}u) \geq \epsilon$, but $d(T^{m_l}u, T^{n_l-1}u) < \epsilon$. Observe that

$$\begin{aligned}\epsilon &\leq d(T^{m_l}u, T^{n_l}u) \leq d(T^{m_l}u, T^{n_l-1}u) + d(T^{n_l-1}u, T^{n_l}u)\\ &\leq d(T^{m_l}u, T^{m_l-1}u) + d(T^{m_l-1}u, T^{n_l}u) + 2d(T^{n_l-1}u, T^{n_l}u)\\ &< d(T^{m_l}u, T^{m_l-1}u) + \epsilon + 2d(T^{n_l-1}u, T^{n_l}u).\end{aligned} \tag{2.12}$$

Since $d(T^n u, T^{n+1}u) \neq 0$, we get

$$\begin{aligned}\lim_{l\to\infty} d(T^{m_l}u, T^{n_l}u) &= \lim_{l\to\infty} d(T^{m_l}u, T^{n_l-1}u) = \lim_{l\to\infty} d(T^{m_l-1}u, T^{n_l-1}u)\\ &= \lim_{l\to\infty} d(T^{m_l-1}u, T^{m_l}u) = \epsilon.\end{aligned} \tag{2.13}$$

Since T is an (α, β, F)-Geraghty type contraction mapping and $\alpha(u, v) \geq 1$, we obtain

$$\begin{aligned}\tau + F(d(T^{m_l-1}u, T^{n_l-1}u)) &\leq \alpha(T^{m_l-1}u, T^{n_l-1}u)(\tau + F(d(T^{m_l-1}u, T^{n_l-1}u)))\\ &\leq \beta(M(T^{m_l-1}u, T^{n_l-1}u))F(M(T^{m_l-1}u, T^{n_l-1}u)),\end{aligned} \tag{2.14}$$

where

$$\begin{aligned}&M(T^{m_l-1}u, T^{n_l-1}u)\\ &= \max\Big\{d(T^{m_l-1}u, T^{m_l}u), d(T^{n_l-1}u, T^{n_l}u),\\ &\qquad \frac{(1 + d(T^{m_l-1}u, T^{m_l}u))d(T^{n_l-1}u, T^{n_l}u)}{1 + d(T^{m_l-1}u, T^{n_l-1}u)}\Big\}.\end{aligned} \tag{2.15}$$

Letting $l \to \infty$ in (2.15) and using (2.13), we obtain

$$\lim_{l\to\infty} M(T^{m_l-1}u, T^{n_l-1}u) = \epsilon. \tag{2.16}$$

Since $\lim_{l\to\infty} \beta(M(T^{m_l-1}u, T^{n_l-1}u)) \leq 1$, we conclude that

$$\tau + F(\epsilon) \leq \beta(\epsilon)F(\epsilon) \leq F(\epsilon), \tag{2.17}$$

a contradiction since $\tau > 0$. Therefore,

$$\lim_{l\to\infty} d(T^{m_l}u, T^{n_l}) = 0. \tag{2.18}$$

It follows that $\{T^n u\}$ is a Cauchy sequence. From T-orbitally complete, there exists $z \in X$ such that $T^n u \to z$ as $n \to \infty$. To show that $Tz = z$, suppose that

$$d(z, Tz) = \lim_{n\to\infty} d(T^n u, Tz) > 0.$$

We have

$$\begin{aligned}\tau + F(d(u_{n+1}u, Tz))) &= \tau + F(d(T^n u, Tz)) \leq \alpha(T^{n-1}u, z)(\tau + F(d(T^n u, Tz))) \\ &\leq \beta(M_T(T^{n-1}u, z))F(M_T(T^{n-1}u, z)),\end{aligned} \tag{2.19}$$

where

$$M_T(T^{n-1}u, z) = \max\left\{d(T^{n-1}u, T^n u), d(z, Tz), \frac{(1 + d(T^{n-1}u, T^n u))d(z, Tz)}{1 + d(T^{n-1}u, z)}\right\}.$$

Letting $n \to \infty$, we get

$$\lim_{i\to\infty} M_T(T^{n-1}u, z) = \max\left\{d(z, z), d(z, Tz), \frac{(1 + d(z, z))d(z, Tz)}{1 + d(z, z)}\right\} = d(z, Tz).$$

Taking the limits as $n \to \infty$ in (2.19), we get

$$F(d(z, Tz)) \leq \beta(d(z, Tz)))F(d(z, Tz)) - \tau \leq F(d(z, Tz)) - \tau,$$

which is a contradiction. Therefore, we obtain $d(z, Tz) = 0$. Similarly, $d(Tz, z) = 0$. That is, $z = Tz$ and the fixed point of T is z. □

Theorem 2.3. *Under all the conditions of Theorem 2.2, we find that z is a unique fixed point of T.*

Proof. From the proof of Theorem 2.2, z is a fixed point of T. Assume, to get a contradiction, that z and w are distinct fixed points of T. By condition (ii) in Theorem 2.2, we get

$$\begin{aligned}\tau + F(d(z, w)) &= \tau + F(d(Tz, Tw)) \\ &\leq \alpha(z, w)(\tau + F(d(Tz, Tw))) \leq \beta(M_T(z, w)F(M_T(z, w),\end{aligned}$$

where

$$M_T(z, w) = \max\left\{d(z, Tz), d(w, Tw), \frac{(1 + d(z, Tw))d(w, Tw)}{1 + d(z, w)}\right\} = d(z, w).$$

Thus

$$\tau + F(d(z, w)) \leq \beta(d(z, w))F(d(z, w)) \leq F(d(z, w)),$$

which is a contradiction since $\tau > 0$. So $z = w$. Hence, T has a unique fixed point. □

Corollary 2.4. *Let (X, d) be a complete dislocated quasi-metric space such that $T : X \to X$ is a self-mapping for all $u, v \in X$, with $\tau > 0$, $d(Tu, Tv) > 0$ and $\beta \in \Omega$,*

$$\tau + F(d(Tu, Tv)) \le \beta(\max\{d(u, Tu), d(v, Tv)\})F(\max\{d(u, Tu), d(v, Tv)\}). \tag{2.20}$$

Then T has a fixed point $z \in X$.

Example 2.5. *Let $X = [0, \infty)$ and a dislocated quasi-metric $d(u, v) = u + v$ for all $u, v \in X$. Let $\beta(t) = \frac{1}{1+t}$ for all $t > 0$. Then $\beta \in \Omega$. Define a mapping $T : X \to X$ and a function $\alpha : X \times X \to [0, \infty)$ by*

$$T(u) = \begin{cases} \frac{u}{5}, & \text{if } u \in [0, 3], \\ 4u, & \text{if } u > 3, \end{cases} \quad \text{and} \quad \alpha(u, v) = \begin{cases} 1 & \text{if } 0 \le u, v \le 3, \\ 0, & \text{otherwise.} \end{cases}$$

Define the function $F : \mathbb{R}^+ \to \mathbb{R}$ by $F(u) = \ln(u)$ for all $u \in \mathbb{R}^+$ and $\tau > 0$. As $u, v \in X$, $\tau = \ln(1.2)$, by taking $u_1 = 3$, we have

case (i): If $0 \le u, v \le 3$, then $\alpha(u, v) = 1$ and

$$\begin{aligned} \alpha(u, v)(\tau + F(d(Tu, Tv))) &= \ln(1.2) + \ln(\frac{u + v}{5}) \\ &\le \frac{\ln(M_T(u, v))}{1 + M_T(u, v)} = \beta(M_T(u, v))F(M_T(u, v)). \end{aligned}$$

Thus $\alpha(u, v)(\tau + F(d(Tu, Tv))) \le \beta(M_T(u, v))F(M_T(u, v))$ for $0 \le u, v \le 3$.

case (ii): If $u \in [0, 3], v > 3$, or $u, v > 3$, then $\alpha(u, v) = 0$ and we have

$$\alpha(u, v)(\tau + F(d(Tu, Tv))) \le \beta(M_T(u, v))F(M_T(u, v)).$$

Hence, all assumptions of Theorems 2.2 and 2.3 are satisfied and so T has the unique fixed point $z = 0$.

Example 2.6. *Let $X = \{\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\} \cup \mathbb{N}\}$ and a dislocated quasi-metric $d(u, v) = |u - v| + u$ for all $u, v \in X$. Let $\beta(t) = \frac{1}{t}$ for all $t > 0$, then $\beta \in \Omega$. Define a mapping $T : X \to X$ and a function $\alpha : X \times X \to [0, \infty)$ by*

$$T(u) = \begin{cases} \frac{9}{u}, & \text{if } u \ge 3, \\ u, & \text{otherwise,} \end{cases} \quad \text{and} \quad \alpha(u, v) = 1 \text{ for all } u, v \in X.$$

Define the function $F : \mathbb{R}^+ \to \mathbb{R}$ *by* $F(u) = \ln(u)$ *for all* $u \in \mathbb{R}^+$ *and* $\tau > 0$. *As* $u, v \in X$, $\tau = \ln(1.2)$, *by taking* $u_1 = 3$, *we have* $\alpha(u, v) = 1$ *and*

$$\alpha(u,v)(\tau + F(d(Tu,Tv))) = \ln(1.2) + \ln(\left|\frac{9}{u} - \frac{9}{v}\right| + \frac{9}{u}) \leq \frac{\ln(M_T(u,v))}{M_T(u,v)} = \beta(M_T(u,v))F(M_T(u,v)).$$

Thus, $\alpha(u,v)(\tau+F(d(Tu,Tv))) \leq \beta(M_T(u,v))F(M_T(u,v))$ *for all* $u, v \in X$.

Hence, all assumptions of Theorems 2.2 and 2.3 are satisfied and so T *has the unique fixed point* $z = 3$.

Acknowledgment. The authors were financially supported by Rajamangala University of Technology Phra Nakhon (RMUTP) Research Scholarship.

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