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On (Λ, p)-extremally disconnected spaces

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Abstract

We introduce the notion of (Λ, p)-extremally disconnected spaces and investigate several characterizations of such spaces.

1 Introduction

The notion of extremally disconnected topological spaces was introduced by Gillman and Jerison [3]. Thompson [9] introduced the notion of S-closed spaces. Niori [7] introduced the concept of locally S-closed spaces which is strictly weaker than that of S-closed spaces. Noiri [6] showed that every locally S-closed weakly Hausdorff space is extremally disconnected. Sivaraj [8] investigated some characterizations of extremally disconnected spaces by utilizing semi-open sets. Noiri [5] investigated several characterizations of extremally disconnected spaces by utilizing preopen sets and semi-preopen sets. Mashhour et al. [4] introduced and investigated the concepts of preopen sets and preclosed sets. Ganster et al. [2] introduced the notions of a pre-Λ-set and a pre-V-set in topological spaces and investigated the fundamental properties of pre-Λ-sets and pre-V-sets. In [1], the present authors

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introduced the notions of (Λ, p)-open sets and (Λ, p)-closed sets which are defined by utilizing the notions of Λ_p-sets and preclosed sets. In this paper, we introduce the concept of (Λ, p)-extremally disconnected spaces and discuss several characterizations of such spaces.

2 Preliminaries

For a subset A of a topological space (X, τ), $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, represent the closure and the interior of A, respectively. A subset A of a topological space (X, τ) is said to be *preopen* [4] if $A \subseteq \mathrm{Int}(\mathrm{Cl}(A))$. The complement of a preopen set is called *preclosed*. The family of all preopen sets of a topological space (X, τ) is denoted by $PO(X, \tau)$. A subset $\Lambda_p(A)$ [2] is defined as follows: $\Lambda_p(A) = \cap\{U \mid A \subseteq U, U \in PO(X, \tau)\}$. A subset A of a topological space (X, τ) is called a Λ_p-*set* [1] (*pre*-Λ-*set* [2]) if $A = \Lambda_p(A)$. A subset A of a topological space (X, τ) is called (Λ, p)-*closed* [1] if $A = T \cap C$, where T is a Λ_p-set and C is a preclosed set. The complement of a (Λ, p)-closed set is called (Λ, p)-*open*. The family of all (Λ, p)-open (resp. (Λ, p)-closed) sets in a topological space (X, τ) is denoted by $\Lambda_p O(X, \tau)$ (resp. $\Lambda_p C(X, \tau)$). Let A be a subset of a topological space (X, τ). A point $x \in X$ is called a (Λ, p)-*cluster point* [1] of A if $A \cap U \neq \emptyset$ for every (Λ, p)-open set U of X containing x. The set of all (Λ, p)-cluster points of A is called the (Λ, p)-*closure* [1] of A and is denoted by $A^{(\Lambda,p)}$. The union of all (Λ, p)-open sets contained in A is called the (Λ, p)-*interior* [1] of A and is denoted by $A_{(\Lambda,p)}$. A subset A of a topological space (X, τ) is said to be $s(\Lambda, p)$-*open* (resp. $p(\Lambda, p)$-*open*, $\beta(\Lambda, p)$-*open*, $r(\Lambda, p)$-*open*) [1] if $A \subseteq [A_{(\Lambda,p)}]^{(\Lambda,p)}$ (resp. $A \subseteq [A^{(\Lambda,p)}]_{(\Lambda,p)}$, $A \subseteq [[A^{(\Lambda,p)}]_{(\Lambda,p)}]^{(\Lambda,p)}$, $A = [A^{(\Lambda,p)}]_{(\Lambda,p)}$). A subset A of a topological space (X, τ) is called $\alpha(\Lambda, p)$-*open* if $A \subseteq [[A_{(\Lambda,p)}]^{(\Lambda,p)}]_{(\Lambda,p)}$.

3 On (Λ, p)-extremally disconnected spaces

We begin this section by introducing the notion of (Λ, p)-extremally disconnected spaces.

Definition 3.1. *A topological space (X, τ) is called (Λ, p)-extremally disconnected if $V^{(\Lambda,p)}$ is (Λ, p)-open in X for every (Λ, p)-open set V of X.*

Theorem 3.2. *For a topological space (X, τ), the following properties are equivalent:*

(1) (X, τ) is (Λ, p)-extremally disconnected.

(2) $A_{(\Lambda,p)}$ is (Λ, p)-closed for every (Λ, p)-closed set A of X.

(3) $[A_{(\Lambda,p)}]^{(\Lambda,p)} \subseteq [A^{(\Lambda,p)}]_{(\Lambda,p)}$ for every subset A of X.

(4) Every $s(\Lambda, p)$-open set is $p(\Lambda, p)$-open.

(5) The (Λ, p)-closure of every $\beta(\Lambda, p)$-open set of X is (Λ, p)-open.

(6) Every $\beta(\Lambda, p)$-open set is $p(\Lambda, p)$-open.

(7) For every subset A of X, A is $\alpha(\Lambda, p)$-open if and only if A is $s(\Lambda, p)$-open.

Proof. (1) $\Rightarrow$ (2): Let A be a (Λ, p)-closed set. Then $X - A$ is (Λ, p)-open, by (1), $[X - A]^{(\Lambda,p)} = X - A_{(\Lambda,p)}$ is (Λ, p)-open. Thus $A_{(\Lambda,p)}$ is (Λ, p)-closed.

(2) $\Rightarrow$ (3): Let A be any subset of X. Then $X - A_{(\Lambda,p)}$ is (Λ, p)-closed, by (2), $[X - A_{(\Lambda,p)}]_{(\Lambda,p)}$ is (Λ, p)-closed and hence $[A_{(\Lambda,p)}]^{(\Lambda,p)}$ is (Λ, p)-open. Thus, $[A_{(\Lambda,p)}]^{(\Lambda,p)} \subseteq [A^{(\Lambda,p)}]_{(\Lambda,p)}$.

(3) $\Rightarrow$ (4): Let A be a $s(\Lambda, p)$-open set. By (3), $A \subseteq [A_{(\Lambda,p)}]^{(\Lambda,sp)} \subseteq [A^{(\Lambda,p)}]_{(\Lambda,p)}$. Thus A is $p(\Lambda, p)$-open.

(4) $\Rightarrow$ (5): Let A be a $\beta(\Lambda, p)$-open set. Then $A^{(\Lambda,p)}$ is $s(\Lambda, p)$-open, by (4), $A^{(\Lambda,p)}$ is $p(\Lambda, p)$-open. Thus $A^{(\Lambda,p)} \subseteq [A^{(\Lambda,p)}]_{(\Lambda,p)}$ and hence $A^{(\Lambda,p)}$ is (Λ, p)-open.

(5) $\Rightarrow$ (6): Let A be a $\beta(\Lambda, p)$-open set. By (5), $A^{(\Lambda,p)} = [A^{(\Lambda,p)}]_{(\Lambda,p)}$. Thus $A \subseteq A^{(\Lambda,p)} = [A^{(\Lambda,p)}]_{(\Lambda,p)}$ and hence A is $p(\Lambda, p)$-open.

(6) $\Rightarrow$ (7): Let A be a $s(\Lambda, p)$-open set. Then A is $\beta(\Lambda, p)$-open, by (6), A is $p(\Lambda, p)$-open. Since A is $s(\Lambda, p)$-open and $p(\Lambda, p)$-open, we have A is $\alpha(\Lambda, p)$-open.

(7) $\Rightarrow$ (1): Let A be a (Λ, p)-open set. Then $A^{(\Lambda,p)}$ is $s(\Lambda, p)$-open, by (7), $A^{(\Lambda,p)}$ is $\alpha(\Lambda, p)$-open. Thus $A^{(\Lambda,p)} \subseteq [[[A^{(\Lambda,p)}]_{(\Lambda,p)}]^{(\Lambda,p)}]_{(\Lambda,p)} = [A^{(\Lambda,p)}]_{(\Lambda,p)}$ and hence $A^{(\Lambda,p)} = [A^{(\Lambda,p)}]_{(\Lambda,p)}$. Therefore, $A^{(\Lambda,p)}$ is (Λ, p)-open. This shows that (X, τ) is (Λ, p)-extremally disconnected. □

Theorem 3.3. *For a topological space (X, τ), the following properties are equivalent:*

(1) (X, τ) is (Λ, p)-extremally disconnected.

(2) For every (Λ, p)-open sets U and V such that $U \cap V = \emptyset$, there exist disjoint (Λ, p)-closed sets F and H such that $U \subseteq F$ and $V \subseteq H$.

(3) $U^{(\Lambda,p)} \cap V^{(\Lambda,p)} = \emptyset$ *for every* (Λ,p)*-open sets* U *and* V *such that* $U \cap V = \emptyset$.

(4) $[[A^{(\Lambda,p)}]_{(\Lambda,p)}]^{(\Lambda,p)} \cap U^{(\Lambda,p)} = \emptyset$ *for every subset* A *of* X *and every* (Λ,p)*-open set* U *such that* $A \cap U = \emptyset$.

Proof. (1) $\Rightarrow$ (2): Let U and V be (Λ,p)-open sets such that $U \cap V = \emptyset$. Then $U^{(\Lambda,p)}$ and $X - U^{(\Lambda,p)}$ are disjoint (Λ,p)-closed sets containing U and V, respectively.

(2) $\Rightarrow$ (3): Let U and V be (Λ,p)-open sets such that $U \cap V = \emptyset$. By (2), there exist disjoint (Λ,p)-closed sets F and H such that $U \subseteq F$ and $V \subseteq H$. Thus $U^{(\Lambda,p)} \cap V^{(\Lambda,p)} \subseteq F \cap H = \emptyset$ and hence $U^{(\Lambda,p)} \cap V^{(\Lambda,p)} = \emptyset$.

(3) $\Rightarrow$ (4): Let A be any subset of X and U be a (Λ,p)-open set such that $A \cap U = \emptyset$. Since $[A^{(\Lambda,p)}]_{(\Lambda,p)}$ is (Λ,sp)-open and $[A^{(\Lambda,p)}]_{(\Lambda,p)} \cap U = \emptyset$, by (3), $[[A^{(\Lambda,p)}]_{(\Lambda,p)}]^{(\Lambda,p)} \cap U^{(\Lambda,p)} = \emptyset$.

(4) $\Rightarrow$ (1): Let U be a (Λ,p)-open set. Then$(X - U^{(\Lambda,p)}) \cap U = \emptyset$. Since $X - U^{(\Lambda,p)}$ is (Λ,p)-open and by (4), $[[U^{(\Lambda,p)}]_{(\Lambda,p)}]^{(\Lambda,p)} \cap [X - U^{(\Lambda,p)}]^{(\Lambda,p)} = \emptyset$. Since U is (Λ,p)-open, we have $U^{(\Lambda,p)} \cap [X - [U^{(\Lambda,p)}]_{(\Lambda,p)}] = \emptyset$ and hence $U^{(\Lambda,p)} \subseteq [U^{(\Lambda,p)}]_{(\Lambda,p)}$. Thus $U^{(\Lambda,p)}$ is (Λ,p)-open. This shows that (X,τ) is (Λ,p)-extremally disconnected. □

Theorem 3.4. *For a topological space* (X,τ)*, the following properties are equivalent:*

(1) (X,τ) *is* (Λ,p)*-extremally disconnected.*

(2) The (Λ,p)*-closure of every* $s(\Lambda,p)$*-open set of* X *is* (Λ,p)*-open.*

(3) The (Λ,p)*-closure of every* $p(\Lambda,p)$*-open set of* X *is* (Λ,p)*-open.*

(4) The (Λ,p)*-closure of every* $r(\Lambda,p)$*-open set of* X *is* (Λ,p)*-open.*

Proof. (1) $\Rightarrow$ (2): Let U be a $s(\Lambda,p)$-open set. Then, U is $\beta(\Lambda,p)$-open, by Theorem 3.2, $U^{(\Lambda,p)}$ is (Λ,p)-open.

(2) $\Rightarrow$ (4): Let U be a $r(\Lambda,p)$-open set. Then, we have U is $s(\Lambda,p)$-open. Thus, by (2), $U^{(\Lambda,p)}$ is (Λ,p)-open.

(4) $\Rightarrow$ (1): Let U be a (Λ,p)-open set. Then $[U^{(\Lambda,p)}]_{(\Lambda,p)}$ is $r(\Lambda,p)$-open and hence $[[U^{(\Lambda,p)}]_{(\Lambda,p)}]^{(\Lambda,p)}$ is (Λ,p)-open. Thus $U^{(\Lambda,p)} \subseteq [[U^{(\Lambda,p)}]_{(\Lambda,p)}]^{(\Lambda,p)} = [[[U^{(\Lambda,p)}]_{(\Lambda,p)}]^{(\Lambda,p)}]_{(\Lambda,p)} = [U^{(\Lambda,p)}]_{(\Lambda,p)}$. Therefore, $U^{(\Lambda,p)}$ is (Λ,p)-open. This shows that (X,τ) is (Λ,p)-extremally disconnected.

(1) $\Rightarrow$ (3): Let U be a $p(\Lambda,p)$-open set. Then U is $\beta(\Lambda,p)$-open. By Theorem 3.2, $U^{(\Lambda,p)}$ is (Λ,p)-open.

(3) $\Rightarrow$ (4): Let U be a $r(\Lambda, p)$-open set. Then U is $p(\Lambda, p)$-open and by (3), $U^{(\Lambda,p)}$ is (Λ, p)-open. □

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