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On the Diophantine Equation $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{n}$

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Abstract

In this paper, we study positive integer solutions of the Diophantine equation $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{n}$. Using elementary methods, we show that if $n=1$, then the equation has only the three solutions $(2,3,6)$, $(2,4,4)$, and $(3,3,3)$. If $n=3$, then the equation has exactly twenty one solutions, which we will write down down explicitly. These results complement the result by Rabago and Tagle [3] who found the positive integer solutions in the case $n=2$.

1 Introduction

In 2013, Rabago and Tagle [3] found all positive integer solutions of the Diophantine equation

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{2}.$$

Later, Sándor [4] offered an elementary approach to the solution of this equation. In 2019, Pakapongpun [2] solved the Diophantine equation

$$\frac{1}{x}+\frac{2}{y}+\frac{3}{z}=\frac{1}{2}.$$

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In 2022, Srimud et al. [5] discovered all positive integer solutions of the Diophantine equation
$$\frac{2}{x}+\frac{3}{y}+\frac{4}{z}=\frac{1}{2}.$$
In the same year, Chinram, Sirikantisophon and Kaewchay [1] found all positive integer solutions of the Diophantine equation
$$\frac{1}{x}+\frac{2}{y}+\frac{3}{z}=\frac{1}{3}.$$

In this paper, we study positive integer solutions of the Diophantine equation
$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{n}, \tag{1.1}$$
where n, x, y and z are positive integers. Without loss of generality, we may assume that $x \leq y \leq z$.

2 Results

Theorem 2.1. *If the Diophantine equation (1.1) has a solution, then* $n+1 \leq x \leq 3n$, $2n+1 \leq y \leq 2n^2+2n$ *and* $3n \leq z \leq n^4+2n^3+2n^2+n$.

Proof. Since $x \leq y \leq z$, we get from (1.1) that $n+1 \leq x \leq 3n$. Then there exists a positive integer a with $1 \leq a \leq 2n$ such that $x = n+a$. Therefore, $y \geq n+a$. From (1.1), we have
$$\frac{1}{y}+\frac{1}{z}=\frac{a}{n(n+a)}. \tag{2.2}$$
Thus
$$\frac{n(n+a)}{a} < y \leq \frac{2n(n+a)}{a} \leq 2n^2+2n.$$
Case 1. $a \leq n$. Then
$$2n = \frac{n(n+n)}{n} \leq \frac{n(n+a)}{a} < y.$$
Case 2. $a > n$. Since $y \geq n+a$, we have $y \geq 2n+1$.
By both cases, we get $y \geq 2n+1$. Now, we consider z. From (1.1) and $x \leq y \leq z$, we have $3n \leq z$. By (2.2), we have
$$\frac{1}{y}+\frac{1}{z} \geq \frac{1}{n(n+1)}$$

and $n(n+1) < y$. It follows that

$$\frac{1}{z} \geq \frac{1}{n(n+1)} - \frac{1}{y} \geq \frac{1}{n(n+1)} - \frac{1}{n(n+1)+1} = \frac{1}{n(n+1)(n^2+n+1)}.$$

Then $z \leq n^4 + 2n^3 + 2n^2 + n$. □

Theorem 2.2. *If $n = 1$, then the Diophantine equation (1.1) has exactly the three positive integer solutions $(x, y, z) \in \{(2, 3, 6), (2, 4, 4), (3, 3, 3)\}$.*

Proof. Since $x \leq y \leq z$ and by (1.1), $2 \leq x \leq 3$.
Case 1. $x = 2$. From (1.1), we get $3 \leq y \leq 4$. If $y = 3$, then we have $z = 6$. Thus $(x, y, z) = (2, 3, 6)$. If $y = 4$, then we have $z = 4$. Therefore, $(x, y, z) = (2, 4, 4)$.
Case 2. $x = 3$. From (1.1), we obtain $y = 3$ and so $z = 3$. Thus $(x, y, z) = (3, 3, 3)$. □

Theorem 2.3. *If $n \geq 2$, then the Diophantine equation (1.1) has at least five positive integer solutions $(x, y, z) \in \{(n+1, n^2+n+1, n^4+2n^3+2n^2+n), (n+1, n^2+2n, n^3+3n^2+2n), (n+1, n^2+2n+1, n^3+2n^2+n), (n+1, 2n^2+n, 2n^2+3n+1), (n+1, 2n^2+2n, 2n^2+2n)\}$.*

Proof. Suppose that $x = n + 1$. From (1.1), we get

$$\frac{1}{y} + \frac{1}{z} = \frac{1}{n(n+1)},$$

and so $(y - n(n+1))(z - n(n+1)) = n^2(n+1)^2$. Since $n \geq 2$, there are five possible cases.
Case 1. $y - n(n+1) = 1$ and $z - n(n+1) = n^2(n+1)^2$. Then $(x, y, z) = (n+1, n^2+n+1, n^4+2n^3+2n^2+n)$.
Case 2. $y - n(n+1) = n$ and $z - n(n+1) = n(n+1)^2$. Then $(x, y, z) = (n+1, n^2+2n, n^3+3n^2+2n)$.
Case 3. $y - n(n+1) = n+1$ and $z - n(n+1) = n^2(n+1)$. Then $(x, y, z) = (n+1, n^2+2n+1, n^3+2n^2+n)$.
Case 4. $y - n(n+1) = n^2$ and $z - n(n+1) = (n+1)^2$. Then $(x, y, z) = (n+1, 2n^2+n, 2n^2+3n+1)$.
Case 5. $y - n(n+1) = n(n+1)$ and $z - n(n+1) = n(n+1)$. Then $(x, y, z) = (n+1, 2n^2+2n, 2n^2+2n)$. □

Theorem 2.4. *If $n = 3$, then the Diophantine equation (1.1) has exactly the twenty one positive integer solutions*

$$(x, y, z) \in \{(4,13,156),(4,14,84),(4,15,60),(4,16,48),(4,18,36),(4,20,30),(4,21,28),(4,24,24),(5,8,120),(5,9,45),(5,10,30),(5,12,20),(5,15,15),(6,7,42),(6,8,24),(6,9,18),(6,10,15),(6,12,12),(7,7,21),(8,8,12),(9,9,9)\}.$$

Proof. By Theorem 2.1, we have $4 \le x \le 9$.
Case 1. $x = 4$. From (1.1), $(y-12)(z-12) = 144$. Since $x \le y \le z$, we have the following subcases.

Subcase 1.1 $y - 12 = 1$ and $z - 12 = 144$. Then $(x, y, z) = (4, 13, 156)$.
Subcase 1.2 $y - 12 = 2$ and $z - 12 = 72$. Then $(x, y, z) = (4, 14, 84)$.
Subcase 1.3 $y - 12 = 3$ and $z - 12 = 48$. Then $(x, y, z) = (4, 15, 60)$.
Subcase 1.4 $y - 12 = 4$ and $z - 12 = 36$. Then $(x, y, z) = (4, 16, 48)$.
Subcase 1.5 $y - 12 = 6$ and $z - 12 = 24$. Then $(x, y, z) = (4, 18, 36)$.
Subcase 1.6 $y - 12 = 8$ and $z - 12 = 18$. Then $(x, y, z) = (4, 20, 30)$.
Subcase 1.7 $y - 12 = 9$ and $z - 12 = 16$. Then $(x, y, z) = (4, 21, 28)$.
Subcase 1.8 $y - 12 = 12$ and $z - 12 = 12$. Then $(x, y, z) = (4, 24, 24)$.

Case 2. $x = 5$. From (1.1), we get

$$\frac{1}{y} + \frac{1}{z} = \frac{2}{15}. \tag{2.3}$$

Since $x \le y \le z$, we have $8 \le y \le 15$.
From (2.3), $(x, y, z) \in \{(5, 8, 120), (5, 9, 45), (5, 10, 30), (5, 12, 20), (5, 15, 15)\}$.
Case 3. $x = 6$. From (1.1), we get

$$\frac{1}{y} + \frac{1}{z} = \frac{1}{6}. \tag{2.4}$$

Since $x \le y \le z$, we have $7 \le y \le 12$.
From (2.4), $(x, y, z) \in \{(6, 7, 42), (6, 8, 24), (6, 9, 18), (6, 10, 15), (6, 12, 12)\}$.
Case 4. $x = 7$. From (1.1), we get

$$\frac{1}{y} + \frac{1}{z} = \frac{4}{21}. \tag{2.5}$$

Since $x \le y \le z$, we have $y = 7$ and so $z = 21$. Then $(x, y, z) = (7, 7, 21)$.
Case 5. $x = 8$. From (1.1), we get

$$\frac{1}{y} + \frac{1}{z} = \frac{5}{24}. \tag{2.6}$$

Since $x \leq y \leq z$, we have $y = 8$ and so $z = 12$. Then $(x, y, z) = (8, 8, 12)$.
Case 6. $x = 9$. From (1.1), we get

$$\frac{1}{y}+\frac{1}{z}=\frac{6}{27}. \tag{2.7}$$

Since $x \leq y \leq z$, we have $y = 9$ and so $z = 9$. Then $(x, y, z) = (9, 9, 9)$. □

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