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An analytical approach to the Boussaïri and Chergui conjecture

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Abstract

The aim of this work is to provide an algorithm returning all matrices having equal corresponding principal minors of all orders to an arbitrary matrix A an $n \times n$ real or complex skew-symmetric matrix using the conjecture of Boussaïri and Chergui. Further results regarding the conjecture are presented and discussed.

1 Introduction

Given a vector of of 2^n values representing the principal minors of $n \times n$ real or complex matrix, the generation of one matrix having the elements of the vector as principal minors in a polynomial time was already solved in the general form. However, the solution is not unique. In this paper, we are interested in answering the question of generating all matrices having equal principal minors of all order to an arbitrary skew symmetric matrix through an algorithm.

2 Definitions and examples

Definition 2.1. *Let A an $n \times n$ real or complex matrix.*
We define a set of transformations $\mathcal{T}$ by the set $\left\{t, t \circ A \overset{pm}{=} A\right\}$, *where* $\circ$

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is the Hadamard product [1] and t is an $n \times n$ real or complex matrix such that for all matrices $t \in \mathcal{T}$, the matrix $t \circ A$ have the same principal minors of all orders. We use the notation $A \overset{pm}{=} t \circ A$ to specify principal minors of all orders equality.
A transformation is said to be elementary, if A and $t \circ A$ are both symmetric or both skew-symmetric accordingly and t having elements in {-1,0,1}.

Remark 2.2. *Some remarks about the previous definition*

1. *Let $A = [t_{ij}]_{1 \leq i,j \leq n}$ be a matrix with some null elements. Clearly, the corresponding elements in all elements of $\mathcal{T}$ can be any numbers $\in \mathbb{C}$. By convention, we chose those elements to be null as well.*
2. *A skew-symmetric transformation element $t \in \mathcal{T}$ switches the form of a symmetric matrix to a skew-symmetric matrix and vice-versa.*
3. *For an elementary transformation $\mathcal{T}$, all the elements $t \in \mathcal{T}$ are symmetric matrices.*
4. *The set of transformations of an arbitrary matrix is included but not always equal to the set of transformations of a specific matrix of the same size (see example 3.3).*

3 Examples

Below are some examples of transformations preserving principal minors.

Example 3.1. *An example of a transformation not preserving the form of the initial matrix. The resulting matrix is neither symmetric nor skew-symmetric.*

$$\begin{pmatrix} 0 & -i & i \\ i & 0 & 1 \\ -i & 1 & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & -x_{12} \cdot i & x_{13} \cdot i \\ -x_{12} \cdot i & 0 & x_{23} \\ x_{13} \cdot i & -x_{23} & 0 \end{pmatrix} \overset{pm}{=} \begin{pmatrix} 0 & x_{12} & x_{13} \\ -x_{12} & 0 & x_{23} \\ -x_{13} & -x_{23} & 0 \end{pmatrix}$$

Example 3.2. *An example of an elementary transformation.*

$$\begin{pmatrix} 0 & 1 & -1 & 1 \\ 1 & 0 & -1 & 1 \\ -1 & -1 & 0 & -1 \\ 1 & 1 & -1 & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & x_{12} & x_{13} & x_{14} \\ -x_{12} & 0 & x_{23} & x_{24} \\ -x_{13} & -x_{23} & 0 & x_{34} \\ -x_{14} & -x_{24} & -x_{34} & 0 \end{pmatrix} \overset{pm}{=} \begin{pmatrix} 0 & x_{12} & -x_{13} & x_{14} \\ -x_{12} & 0 & -x_{23} & x_{24} \\ x_{13} & x_{23} & 0 & -x_{34} \\ -x_{14} & -x_{24} & x_{34} & 0 \end{pmatrix}$$

Example 3.3. *An example of an elementary transformation switching a single element's sign for a specific matrix. Such transformation is not valid for arbitrary matrices.*

$$\begin{pmatrix} 0 & -1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & -1 & 1 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 & 1 & -1 \\ -1 & -1 & 0 & 1 & 1 & -1 \\ -1 & -1 & -1 & 0 & 1 & -1 \\ -1 & -1 & -1 & -1 & 0 & -1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} \overset{pm}{=} \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & -1 \\ -1 & 0 & 1 & 1 & 1 & -1 \\ -1 & -1 & 0 & 1 & 1 & -1 \\ -1 & -1 & -1 & 0 & 1 & -1 \\ -1 & -1 & -1 & -1 & 0 & -1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

4 The BC conjecture

In order to state the *BC conjecture* named after Boussaïri and Chergui [6] specifying an equivalence relation between skew-symmetric matrices having equal corresponding principal minors, we need the following definitions and notations. Let $A = [a_{ij}]$ be an $n \times n$ matrix and let X; Y be two nonempty subsets of [n] (where [n] := 1,..,n). We denote by A[X; Y] the sub-matrix of A having row indices in X and column indices in Y . If X = Y, then A[X;X] is a principal sub-matrix of A and we abbreviate this as A[X].

Following [6], a subset X of [n] is an HL-clan of A if both of matrices $A[\bar{X}; X]$ and $A[X; \bar{X}]$ have rank of at most 1 (where $\bar{X} := [n] \backslash X$). By definition, $\emptyset$, $[n]$ and singletons are HL-clans. Considering the particular case when A is skew-symmetric, let X be a subset of [n]. We denote by $Inv(X; A) := [t_{ij}]$ the matrix obtained from A as follows. For any $i; j \in [n]$, $t_{ij} = -a_{ij}$ if $i; j \in X$ and $t_{ij} = a_{ij}$, otherwise. More generally, let A and B be two skew-symmetric matrices, assume that there exists a sequence $A_0 = A, ..., A_m = B$ of $n \times n$ skew-symmetric matrices such that for $k = 0, ..., m$; $A_{k+1} = Inv(X_k; A_k)$, where X_k is an HL-clan of A_k. Two matrices A,B obtained in this way are called HL-clan-reversal-equivalent.
The conjecture of Boussaïri and Chergui is stated as follows:

4.1 Conjecture

Two $n \times n$ skew-symmetric real matrices have equal corresponding principal minors of all orders if and only if they are HL-clan-reversal-equivalent.

5 Determining the principal minors of skew-symmetric matrices

For dense symmetric matrices, Oeding [3] pointed that the principal minors of order at most 3 define the rest of the principal minors, but since the odd principal minors of skew-symmetric matrices are null, we need to use the fourth principal minors. In the following, we define an equivalence relation between *pairwise different* matrices having equal corresponding principal minors and the transformation results of the program in [7.1]. The pairwise different condition is specified to preserve the general form of matrices.

6 Main Theorem

For two dense skew-symmetric $n \times n$ matrices pairwise different, the following statements are equivalent:

1. A and B have equal corresponding principal minors of all orders;
2. A and B have equal corresponding principal minors of order at most 4;
3. An elementary transformation t is returned by the SMPM program [7.1] such that $A \overset{pm}{=} t \circ B$.

6.1 Proposition

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two 4x4 dense skew-symmetric matrices such $a_{ij} = \pm b_{ij}, i, j \in \{1, .., 4\}$ have the same determinant if and only if one of the above statements is true:

1. The couple's elements $\{a_{12}a_{34}, b_{12}b_{34}\}$, $\{a_{13}a_{24}, b_{13}, b_{24}\}$ and $\{a_{23}a_{14}nb_{23}, b_{14}\}$ have respectively and independently a positive or negative but the same signs;
2. The couple's elements $\{a_{12}a_{34}, b_{12}b_{34}\}$, $\{a_{13}a_{24}, b_{13}, b_{24}\}$ and $\{a_{23}a_{14}nb_{23}, b_{14}\}$ have all respectively the opposite sign;
3. The determinant is null;
4. At least one couple from the set $\{a_{12}a_{34}, a_{13}a_{24}, -a_{23}a_{14}\}$ are equal. In that case, the matrices A and B are said to be *pairwise equal;* otherwise, they are *pairwise different.*

Definition 6.1. *A skew-symmetric matrix A such*

$$A = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix}$$

is said pairwise different if the scalars af, $-be$ and dc are pairwise different.

6.2 Proof

Let A and B be two dense skew-symmetric matrices such that

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & b_{12} & b_{13} & b_{14} \\ -b_{12} & 0 & b_{23} & b_{24} \\ -b_{13} & -b_{23} & 0 & b_{34} \\ -b_{14} & -b_{24} & -b_{34} & 0 \end{pmatrix}.$$

Since $det(A) = (a_{12}a_{34} - a_{13}a_{24} + a_{23}a_{14})^2$, $det(B) = (b_{12}b_{34} - b_{13}b_{24} + b_{23}b_{14})^2$, and $a_{ij} = \pm b_{ij}, i, j \in \{1, .., 4\}$ and since there are no zeros off the diagonal and $a_{ij} = \pm b_{ij}$, there exists three unknowns $\{x, y, z\}$ having values in $\{-1, 1\}$ such $det(B) = (xa_{14}a_{23} - ya_{13}a_{24} + za_{12}a_{34})^2$. We can easily check that solving $det(A) = det(B)$ leads to the previous statements.

6.3 Remark

As a consequence of proposition 6.1, the elementary transformations for 4×4 pairwise different matrices are :

1. $\begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{pmatrix}$
2. $\begin{pmatrix} 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{pmatrix}$
3. $\begin{pmatrix} 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ -1 & 1 & -1 & 0 \end{pmatrix}$
4. $\begin{pmatrix} 0 & 1 & 1 & -1 \\ -1 & 0 & -1 & 1 \\ -1 & 1 & 0 & 1 \\ 1 & -1 & -1 & 0 \end{pmatrix}$
5. $\begin{pmatrix} 0 & -1 & -1 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & -1 \\ -1 & 1 & 1 & 0 \end{pmatrix}$
6. $\begin{pmatrix} 0 & -1 & 1 & -1 \\ 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{pmatrix}$
7. $\begin{pmatrix} 0 & 1 & -1 & -1 \\ -1 & 0 & -1 & -1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix}$
8. $\begin{pmatrix} 0 & -1 & -1 & -1 \\ 1 & 0 & -1 & -1 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$
9. $\begin{pmatrix} 0 & -1 & -1 & 1 \\ 1 & 0 & -1 & 1 \\ 1 & 1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{pmatrix}$
10. $\begin{pmatrix} 0 & 1 & 1 & -1 \\ -1 & 0 & 1 & -1 \\ -1 & -1 & 0 & -1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$
11. $\begin{pmatrix} 0 & -1 & -1 & -1 \\ 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & -1 & 0 \end{pmatrix}$
12. $\begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & -1 & -1 \\ -1 & 1 & 0 & -1 \\ -1 & 1 & 1 & 0 \end{pmatrix}$
13. $\begin{pmatrix} 0 & -1 & 1 & 1 \\ 1 & 0 & -1 & -1 \\ -1 & 1 & 0 & 1 \\ -1 & 1 & -1 & 0 \end{pmatrix}$
14. $\begin{pmatrix} 0 & 1 & -1 & -1 \\ -1 & 0 & 1 & 1 \\ 1 & -1 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{pmatrix}$
15. $\begin{pmatrix} 0 & 1 & -1 & 1 \\ -1 & 0 & -1 & 1 \\ 1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{pmatrix}$
16. $\begin{pmatrix} 0 & -1 & 1 & -1 \\ 1 & 0 & 1 & -1 \\ -1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix}$

6.4 Lemma

Let A and B be two 6×6 dense skew-symmetric matrices equals up to a sign pairwise different. A and B have equal corresponding principal minors of order at most 4 if and only if they have equal corresponding principal minors of all orders.

6.5 Proof

This result is proved under weaker conditions by Boussaïri and Chergui [6].

6.6 Definition

Let A be a skew-symmetric matrix and let v be a 2^{n-1} vector representing the set of its principal minors. v is said to be in *combinatorial order* if it is ordered by blocks of principal minors order, then by the combinatorial order of chosen rows and columns for each order. In other words, for a specific principal minor order k, there are $c = \binom{n}{k}$ vectors of size k representing the selected rows/columns forming the principal sub-matrix, such the rows/columns number of the vectors are in ascending order in the vectors individually and for the corresponding order in the set.

6.7 Remark

As a consequence of the definition of combinatorial order, the entries of v for a 3×3 matrix are as follows:

$$v = \left(\underbrace{\underbrace{A[1]}_{PM1_1}, \underbrace{A[2]}_{PM1_2}, \underbrace{A[3]}_{PM1_3}}_{\text{PM1}}, \underbrace{\underbrace{A[1,2]}_{PM2_1}, \underbrace{A[1,3]}_{PM2_2}, \underbrace{A[2,3]}_{PM2_3}}_{\text{PM2}}, \underbrace{\underbrace{A[1,2,3]}_{PM3_1}}_{\text{PM3}} \right)$$

6.8 Proof of the main Theorem

The equivalence between the second and last statements holds by design, the proof of the equivalence between the first and second statements are based on lemma 6.4.

7 The algorithm and computational results

Throughout this section, we explain the algorithm of the program SMPM generating the elementary transformations for an $n \times n$ arbitrary dense matrix. We then present the computational results and their analysis.

7.1 The program SMPM

Using the results of the proposition 6.1, we initiate the algorithm with the matrices obtained in 6.3. The matrices can be hard coded in the implementation because this same set is used abstraction of the size of the transformations. The algorithm uses the principle of theorem 6 by building recursively matrices of equal corresponding principal minors of $4th$ order. The algorithm proceeds as follows:

Algorithm 1: Generate Elementary transformations of an $n \times n$ skew-symmetric matrix

Result: Generate Elementary transformations of an $n \times n$ skew-symmetric matrix
initialization of $\mathcal{E}_{1 \leq j \leq 16}$ `// the set of 16 elementary transformation matrices of order 4`
initialization of $\mathcal{R}_{j=1}$ `// Initiate the result set with one element formed by unknowns elements`
initialization of $PM_{1 \leq i \leq 2^n - 1}$ `// Initiate the set of the principal minors`
for *All principal minors combinations* PM_i **do**
    for *All* $\mathcal{R}_j$ *elements of* $\mathcal{R}$ **do**
        for *All* $\mathcal{E}_k$ *elements of* $\mathcal{E}$ **do**
            if *the elements of* $\mathcal{R}_i$ *are still unknown* **then**
                Fill the unknown elements and add the new matrix to $\mathcal{R}$;
            else if *no element of* E *is equal to* PM_i **then**
                Remove the matrix from $\mathcal{R}$;
        end
    end
end
Return $\mathcal{R}$;

7.2 Elementary transformations for 6×6 matrices

For the case of a 6×6 matrices, the algorithm will return 64 skew-symmetric dense matrices. The Hadamard products of any of those elementary transformations matrix by the resulting matrices share the same principal minors of all orders.

We illustrate all HL-clans relations by the following figure where every point represents a matrix and a line between two matrices represents an HL-Clan relation:

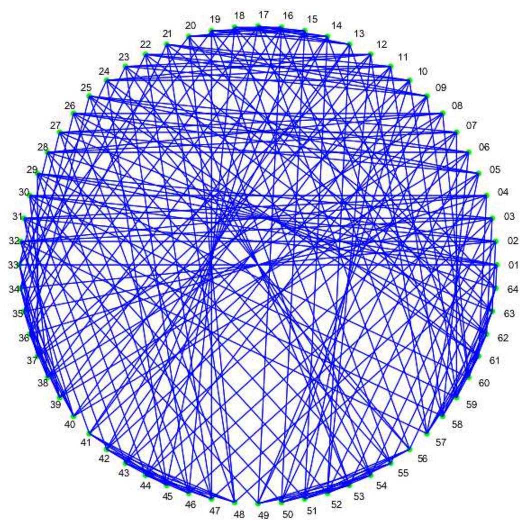

Figure 1: The HL-Clans relations

Therefore, to validate the conjecture, we must find a path between every couple of matrices. More simply, find a path from every matrix to the first one which we present in the following figures for the 6^{th}, 8^{th} and the 10^{th} matrix orders:

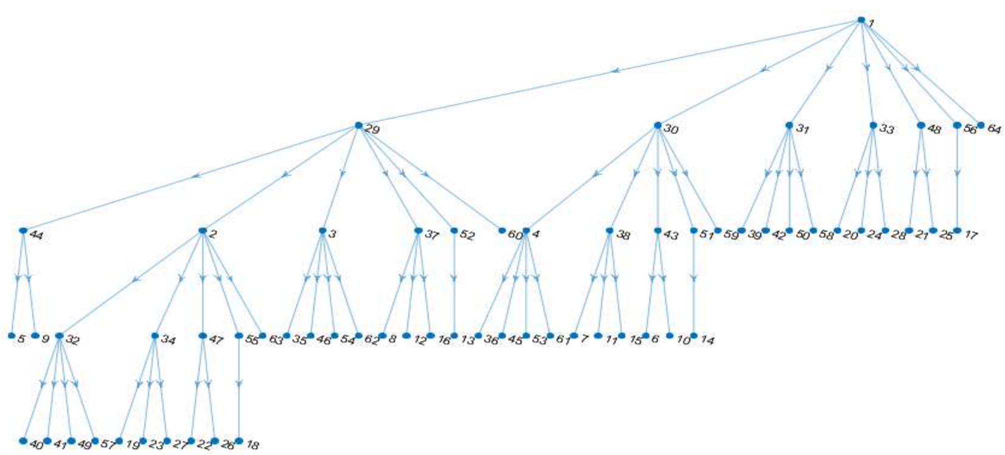

Figure 2: The HL-Clans relations paths from all matrices to the first one for the 6^{th} order matrices

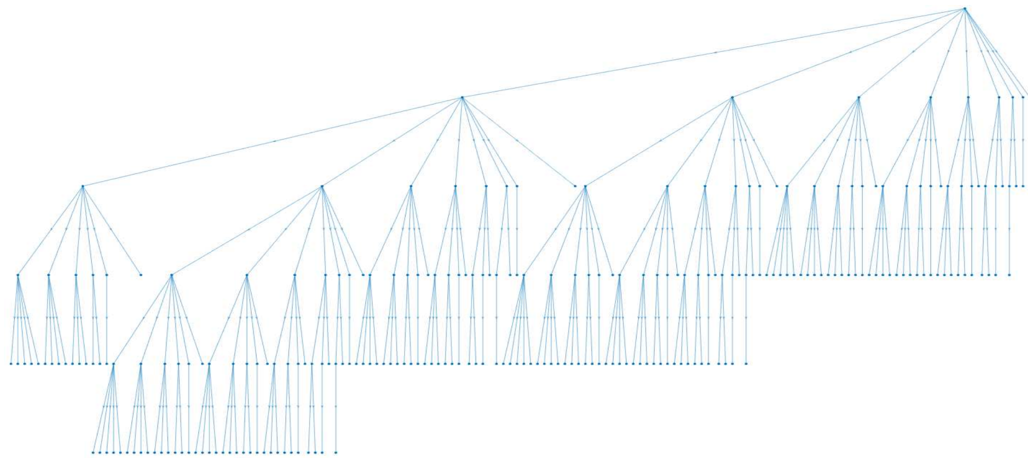

Figure 3: The HL-Clans relations paths from all matrices to the first one for the 8^{th} order matrices

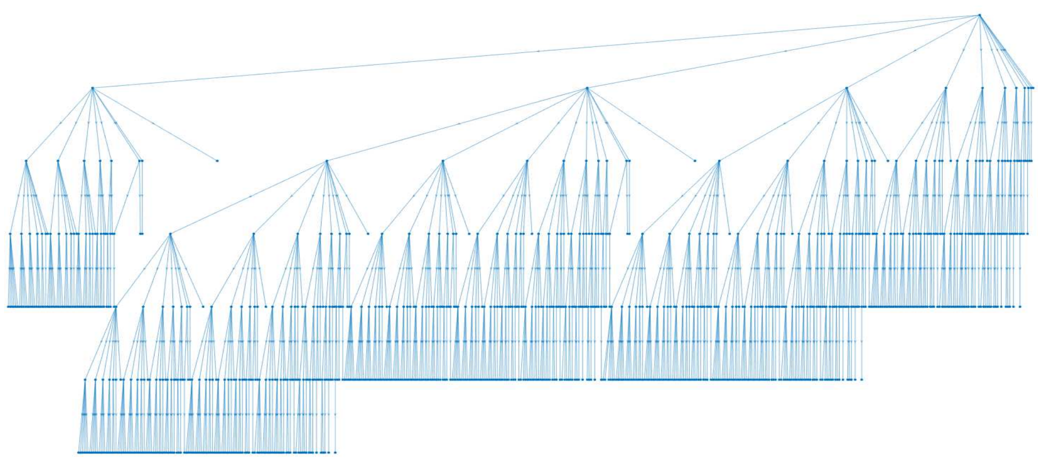

Figure 4: The HL-Clans relations paths from all matrices to the first one for the 10^{th} order matrices

7.3 Computational results

We illustrate the results obtained from the recursive execution of the program for different matrix orders:

Table 1: Execution results

Matrix order	Number of elementary transformations	Number of HL-Clans relations	Maximum path length
6	64	448	4
8	256	2304	5
10	1024	11264	6
12	4096	53248	7
14	16384	245760	8

We can generalize the following results to a skew-symmetric matrix of order n:

1. The number of elementary matrices is 2^n;
2. The number of HL-Clans relations is $(n+1)2^n$;
3. Te Maximum path length is $\frac{n}{2}+1$.

7.4 Practicals examples

Let A and B be two skew-symmetric matrices formed as follows:

$$A = \begin{pmatrix} 0 & 2 & 3 & 5 & 7 & 11 \\ -2 & 0 & 13 & 17 & 19 & 23 \\ -3 & -13 & 0 & 29 & 31 & 37 \\ -5 & -17 & -29 & 0 & 41 & 43 \\ -7 & -19 & -31 & -41 & 0 & 47 \\ -11 & -23 & -37 & -43 & 47 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & -2 & -3 & 5 & 7 & -11 \\ 2 & 0 & 13 & -17 & -19 & 23 \\ 3 & -13 & 0 & -29 & -31 & 37 \\ -5 & 17 & -29 & 0 & 41 & -43 \\ -7 & 19 & -31 & -41 & 0 & -47 \\ 11 & -23 & -37 & 43 & 47 & 0 \end{pmatrix}$$

A and B are formed to be pairwise different. We can identify the corresponding elementary transformations from the result of the SMPM program in [7.2] to the first and eighteenth matrices. Therefore, from the matrices path figure 2 in 7.2, we can deduce a valid path going trough the matrices obtained by the elementary transformations numbered by 29, 2, 55, 18 explained in the following steps.

Table 2: HL-Clan relation steps.

Step	Matrix	HL-Clan
1	$\begin{pmatrix} 0 & 2 & 3 & 5 & 7 & 11 \\ -2 & 0 & 13 & 17 & 19 & 23 \\ -3 & -13 & 0 & 29 & 31 & 37 \\ -5 & -17 & -29 & 0 & 41 & 43 \\ -7 & -19 & -31 & -41 & 0 & 47 \\ -11 & -23 & -37 & -43 & -47 & 0 \end{pmatrix}$	[1,2,3,4,5,6]
2	$\begin{pmatrix} 0 & -2 & -3 & -5 & -7 & -11 \\ 2 & 0 & -13 & -17 & -19 & -23 \\ 3 & 13 & 0 & -29 & -31 & -37 \\ 5 & 17 & 29 & 0 & -41 & -43 \\ 7 & 19 & 31 & 41 & 0 & -47 \\ 11 & 23 & 37 & 43 & 47 & 0 \end{pmatrix}$	[1,2,3,4,5]
3	$\begin{pmatrix} 0 & 2 & 3 & 5 & 7 & -11 \\ -2 & 0 & 13 & 17 & 19 & -23 \\ -3 & -13 & 0 & 29 & 31 & -37 \\ -5 & -17 & -29 & 0 & 41 & -43 \\ -7 & -19 & -31 & -41 & 0 & -47 \\ 11 & 23 & 37 & 43 & 47 & 0 \end{pmatrix}$	[1,3,4,5,6]
4	$\begin{pmatrix} 0 & 2 & -3 & -5 & -7 & 11 \\ -2 & 0 & 13 & 17 & 19 & -23 \\ 3 & -13 & 0 & -29 & -31 & 37 \\ 5 & -17 & 29 & 0 & -41 & 43 \\ 7 & -19 & 31 & 41 & 0 & 47 \\ -11 & 23 & -37 & -43 & -47 & 0 \end{pmatrix}$	[1,2,4,5,6]
5	$\begin{pmatrix} 0 & -2 & -3 & 5 & 7 & -11 \\ 2 & 0 & 13 & -17 & -19 & 23 \\ 3 & -13 & 0 & -29 & -31 & 37 \\ -5 & 17 & 29 & 0 & 41 & -43 \\ -7 & 19 & 31 & -41 & 0 & -47 \\ 11 & -23 & -37 & 43 & 47 & 0 \end{pmatrix}$	

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