International Journal of Mathematics and
Computer Science, **18**(2023), no. 2, 321–333

The Frobenius problems for Sexy Prime Triplets

WonTae Hwang[1], Kyunghwan Song[2]

[1]Department of Mathematics and Institute of Pure and Applied Mathematics
Jeonbuk National University
Baekje-daero, Deokjin-gu, Jeonju-si, Jeollabuk-do, 54896 Republic of Korea

[2]Department of Mathematics
Jeju National University
Jeju, Jeju-si, Republic of Korea

email: khsong@jejunu.ac.kr

(Received December 1, 2022, Accepted January 5, 2023,
Published January 23, 2023)

Abstract

The greatest integer that does not belong to a numerical semigroup S is called the Frobenius number of S. The Frobenius problem, which is also called the coin problem or the money changing problem, is a mathematical problem of finding the Frobenius number. In this paper, we solve the Frobenius problem for sexy prime triplets.

1 Introduction

The greatest integer that does not belong to a numerical semigroup S is called the *Frobenius number* of S and is denoted by $F(S)$. In other words, the Frobenius number is the largest integer that cannot be expressed as a sum $\sum_{i=1}^{n} t_i a_i$, where $t_1, t_2, ..., t_n$ are nonnegative integers and $a_1, a_2, ..., a_n$ are generators of S (See §2.1 for the definition of generators). Finding the

Key words and phrases: Frobenius problem, Sexy primes.
AMS (MOS) Subject Classifications: 11D07, 20M14.
ISSN 1814-0432, 2023, http://ijmcs.future-in-tech.net

Frobenius number is called the *Frobenius problem*, the *coin problem* or the *money changing problem*. The Frobenius problem is not only interesting for pure mathematicians, but it is also connected with graph theory [10, 11] and the theory of computer science [17], as introduced in [16]. There are explicit formulas for the Frobenius number when only two relatively prime numbers are present as a generator of a numerical semigroup [32]. For the case of three relatively prime numbers, it was shown decades ago that there is a somewhat algorithmic method to obtain the Frobenius number using the Euclidean algorithm [22], and more recently, semi-explicit formulas [20, 33] were obtained.

On the other hand, F. Curtis proved in [4] that the Frobenius number for three or more relatively prime numbers cannot be given by a finite set of polynomials, and Ramírez-Alfonsín proved in [18] that the problem is NP-hard. Currently, only algorithmic methods of determining the general formula for the Frobenius number of a numerical semigroup whose generating set has three or more relatively prime numbers [2, 3]. From an algebraic viewpoint, rather than finding the general formula for the case of three or more relatively prime numbers, the formulae for special cases were found such as the Frobenius number of a numerical semigroup whose generating set consists of a geometric sequence [15], Pythagorean triples [6] and three consecutive squares or cubes [12]. Recently, various methods of solving the Frobenius problem for numerical semigroups have been suggested in [1, 23, 27, 28], etc. In particular, a method of computing the Apéry set and obtaining the Frobenius number using the Apéry set is an efficient tool for solving the Frobenius problem for numerical semigroups as reported in [13, 19, 27]. Furthermore, this method was used to obtain the Frobenius number in recent articles such as the ones presenting the Frobenius problems for Fibonacci numerical semigroups [14], Mersenne numerical semigroups [26], Thabit numerical semigroups [24], some generalizations of Thabit numerical semigroups [7, 30, 31], and repunit numerical semigroups [25].

As a motivation of this paper, note that there is a newest result which uses the Apéry set to compute the Frobenius number, genus, pseudo-Frobenius numbers and type of a numerical semigroup whose generating set is a prime k-tuplet [21]. Here, a prime k-tuplet is a sequence of consecutive prime numbers $P_k = (p_1, \ldots, p_k)$, and assuming the generalized Hardy-Littlewood conjecture [8, 9] the number of prime k-tuplets is infinite. Also, in [21], the authors introduce the cousin prime $(p, p+4)$ where p and $p+4$ are primes, sexy prime $(p, p+6)$ where p and $p+6$ are primes, and sexy prime triplet $(p, p+6, p+12)$ where p, $p+6$, and $p+12$ are primes.

In this paper, we find the Apéry set, Frobenius number, pseudo-Frobenius numbers, genus, and type of the semigroups generated by the sexy prime triplets. One of our main results is summarized in the following theorem.

Theorem 1.1. *The Frobenius number of a numerical semigroup with a sexy prime triplet* $\{p, p+6, p+12\}$ *as a generating set is a quadratic polynomial of* p *whose leading coefficient has denominator* 2*.*

For the details, see the proof of Corollary 3.3.

This paper is organized as follows. In Section 2, we introduce some preliminaries on the Frobenius numbers of numerical semigroups with relevant facts (see §2.1), and the sexy prime triplets (see §2.2). In Section 3, we present a method to obtain the Apéry set, Frobenius number, pseudo-Frobenius numbers, genus and type for the semigroups generated by the sexy prime triplets.

2 Preliminaries

In this section, we provide some auxiliary facts for our main results.

2.1 Numerical semigroup and submonoid

Let $\mathbb{N}$ be the set of nonnegative integers. We introduce the notions of a numerical semigroup and a submonoid generated by a nonempty subset.

Definition 2.1. *A* numerical semigroup *is a subset* S *of* $\mathbb{N}$ *that is closed under addition and contains* 0*, such that* $\mathbb{N} \setminus S$ *is finite.*

Definition 2.2. *Given a nonempty subset* A *of a numerical semigroup* $\mathbb{N}$*, we denote by* $\langle A \rangle$ *the submonoid of* $(\mathbb{N}, +)$ *generated by* A*, that is,*

$$\langle A \rangle = \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid n \in \mathbb{N} \setminus \{0\}, a_i \in A, \lambda_i \in \mathbb{N}$$

for all $i \in \{1, \cdots, n\}\}$.

In addition, we recall several theorems and definitions that are directly related to the above concepts.

Theorem 2.3. *([24, 27]). Let* $\langle A \rangle$ *be the submonoid of* $(\mathbb{N}, +)$ *generated by a nonempty subset* $A \subseteq \mathbb{N}$*. Then* $\langle A \rangle$ *is a numerical semigroup if and only if* $\gcd(A) = 1$*.*

One easy thing to see is that if A_1, A_2 are two nonempty subsets of $\mathbb{N}$ with $A_1 \subseteq A_2$, then $\langle A_1 \rangle \subseteq \langle A_2 \rangle$. In particular, we can think of the smallest element in the set of nonempty subsets $A \subseteq \mathbb{N}$ which generate the same numerical semigroup.

Definition 2.4. *If S is a numerical semigroup and $S = \langle A \rangle$, then we say that A is a* system of generators of S. *Moreover, if $S \neq \langle X \rangle$ for all $X \subsetneq A$, then we say that A is a* minimal system of generators of S.

Regarding Definition 2.4, one interesting thing is the following theorem.

Theorem 2.5. *([27]). Every numerical semigroup S admits a finite and unique minimal system of generators.*

This fact naturally motivates:

Definition 2.6. *The cardinality of the minimal system of generators of S is called the* embedding dimension *of S and is denoted by* $\mathrm{e}(S)$.

Now, we provide two more relevant concepts.

Definition 2.7. *The cardinality of $\mathbb{N} \setminus S$ is called the* genus of S *and is denoted by $g(S)$.*

Definition 2.8. *(a) An integer x is a* pseudo-Frobenius number *for S if $x \notin S$ and $x + s \in S$ for all $s \in S \setminus \{0\}$.*
(b) The set of pseudo-Frobenius numbers of S is denoted by $PF(S)$.
(c) The cardinality of $PF(S)$ is called the type *of S and denoted by $t(S)$.*

Example 2.9. *Let $A = \{7, 11, 13\}$. Then we have the following:*

- $S = \langle A \rangle$ *is a numerical semigroup because* $\gcd(7, 11, 13) = 1$.
- *Since* $13 < 2 \cdot 7$, *A is a minimal system of generators of S. In particular, we have* $\mathrm{e}(S) = 3$.
- *Let $a \in \mathbb{N}$. We use a classical method to find the Frobenius number, the genus, the pseudo-Frobenius numbers, and the type of S.*
  - *If $n \equiv 0 \pmod 7$ any positive number $n = 7a$ can be represented.*
  - *If $n \equiv 1 \pmod 7$ any positive number $n = 2 \cdot 11 + 7a = 22 + 7a$ can be represented.*
  - *If $n \equiv 2 \pmod 7$ any positive number $n = 11 + 2 \cdot 13 + 7a = 37 + 7a$ can be represented.*

- *If* $n \equiv 3 \pmod 7$ *any positive number* $n = 11 + 13 + 7a = 24 + 7a$ *can be represented.*
- *If* $n \equiv 4 \pmod 7$ *any positive number* $n = 11 + 7a$ *can be represented.*
- *If* $n \equiv 5 \pmod 7$ *any positive number* $n = 2 \cdot 13 + 7a = 26 + 7a$ *can be represented.*
- *If* $n \equiv 6 \pmod 7$ *any positive number* $n = 13 + 7a$ *can be represented.*

Since any positive number greater than or equal to 31 *can be represented by an integral combination of* 7, 11, *and* 13*, we conclude that* $F(S) = 30$*. Also, since*

$$S = \mathbb{N} \setminus (\{1,2,3,4,5,6\} \cup \{8,15\} \cup \{9,16,23,30\} \cup \{10,17\} \cup \{12,19\}),$$

we have $g(S) = 6 + 2 + 4 + 2 + 2 = 16$*. Finally, it is clear that* $PF(S) = \{15, 30, 19\}$ *and* $t(S) = 3$*.*

For the rest of this subsection, we introduce the method of finding the Frobenius number using Apéry set. We start with the following definition.

Definition 2.10. *Let* S *be a numerical semigroup and let* $x \in S \setminus \{0\}$*. Then we define* the Apéry set of x in S *to be the set* $Ap(S, x) = \{s \in S | s - x \notin S\}$*.*

The relation among the Frobenius number, genus, and Apéry set of a numerical semigroup is given in the following lemma.

Lemma 2.11. ([27, 29])*. Let* S *be a numerical semigroup and let* $x \in S \setminus \{0\}$*. Then we have*
(a) $F(S) = \max(Ap(S, x)) - x$*.*
(b) $g(S) = \frac{1}{x}(\sum_{w \in Ap(S,x)} w) - \frac{x-1}{2}$*.*

Example 2.12. *Let* $S = \langle\{7, 11, 13\}\rangle$ *as in Example 2.9. Then we have* $Ap(S, 7) = \{0, 11, 13, 22, 24, 26, 37\}$*,* $F(S) = \max Ap(S, 7) - 7 = 30$*, and* $g(S) = \frac{1}{7}(0 + 11 + 13 + 22 + 24 + 26 + 37) - \frac{7-1}{2} = 16$*.*

2.2 Sexy prime triplets

Now, we turn our attention to sexy prime triplets.

Definition 2.13. *Sexy primes are pairs of primes of the form* $(p, p + 6)$ *and sexy prime triplets are 3-tuples of primes of the form* $(p, p + 6, p + 12)$*.*

It is clear that if $p+6$ or $p+12$ is not a prime, then $(p, p+6, p+12)$ is not a sexy prime triplet. Thus it is efficient to exclude the trivial cases that $p+6$ or $p+12$ is not a prime. For convenience, we let $p = 60k + \alpha$ where $0 \leq \alpha, \leq 59$. Note that $p+6 = 60k + (\alpha + 6)$ (resp. $p+12 = 60k + (\alpha + 12)$) is not a prime when $\gcd(60, \alpha + 6) \neq 1$ (resp. $\gcd(60, \alpha + 12) \neq 1$) and this condition is equivalent to $\gcd(30, \alpha + 6) \neq 1$ (resp. $\gcd(30, \alpha + 12) \neq 1$).

Now we can tell what the sexy prime triplets look like.

Lemma 2.14. *Let $(p, p+6, p+12)$ be a sexy prime triplet. Then $(p, p+6, p+12)$ is of one of the following 4 forms:*

1. $(30k+1, 30k+7, 30k+13)$
2. $(30k+7, 30k+13, 30k+19)$
3. $(30k+11, 30k+17, 30k+23)$
4. $(30k+17, 30k+23, 30k+29)$

Remark 2.15. *In light of Dickson's conjecture ([5]), we expect that the number of each form of sexy prime triplets is infinite.*

3 Main Results

In this section, we provide our main results on the Frobenius numbers of semigroups generated by each of the form of sexy prime triplets. We give a detailed proof for the semigroup generated by a sexy prime triplet of the form $(30k+1, 30k+7, 30k+13)$ as the first case, and omit the details of the proofs for remaining cases due to the fact that the arguments are somewhat similar as in the first case.

3.1 The first form: $S = \langle 30k+1, 30k+7, 30k+13 \rangle$

We begin with the following easy observation: for any $k \in \mathbb{N}$, we have

1. $(15k+7)(30k+1) = (30k+7) + 15k(30k+13)$
2. $2(30k+7) = (30k+1) + (30k+13)$
3. $(15k+1)(30k+13) = (15k+6)(30k+1) + (30k+7)$

These three equations are the representations of $x(30k+1), y(30k+7), z(30k+13)$ with the least positive integer coefficients x, y, z by the nonnegative integer combinations of other elements in the set $\{30k+1, 30k+7, 30k+13\}$.

Using the above equations, we have the following lemma which indicates the shape of a subset of the Apéry set, $\mathrm{Ap}(S, 30k+1)$.

Lemma 3.1. *Let $k \in \mathbb{N}$ and $S = \langle 30k+1, 30k+7, 30k+13\rangle$. Then the set*

$$\{a(30k+13) \mid a \in \{0,1,\ldots,15k\}\} \cup \{(30k+7)+a(30k+13) \mid a \in \{0,1,\ldots,15k-1\}\}$$

is a subset of $Ap(S, 30k+1)$.

Proof. Since $\gcd(30k+1, 30k+13) = 1$, it is clear that $15k(30k+13) \in \mathrm{Ap}(S, 30k+1)$.

Now, suppose that there is an element $\alpha(30k+7)+\beta(30k+13) \in \{a(30k+13) \mid a \in \{0,1,\ldots,15k-1\}\} \cup \{(30k+7)+a(30k+13) \mid a \in \{0,1,\ldots,15k-1\}\}$ such that $\alpha(30k+7)+\beta(30k+13) \notin \mathrm{Ap}(S, 30k+1)$. Then we have

$$(x_1+1)(30k+1) = (\alpha - x_2)(30k+7) + (\beta - x_3)(30k+13)$$

for some $x_1, x_2, x_3 \in \mathbb{N}$, which, in turn, implies that $\beta \geq 15k$. Therefore $\alpha(30k+7)+\beta(30k+13) \in \mathrm{Ap}(S, 30k+1)$ for any $\alpha(30k+7)+\beta(30k+13) \in \{a(30k+13) \mid a \in \{0,1,\ldots,15k-1\}\} \cup \{(30k+7)+a(30k+13) \mid a \in \{0,1,\ldots,15k-1\}\}$. □

Now we are ready to suggest the following theorem for Apéry set of the sexy prime triplets of the form $(30k+1, 30k+7, 30k+13)$.

Theorem 3.2. *Let $k \in \mathbb{N}$ and $S = \langle 30k+1, 30k+7, 30k+13\rangle$. Then we have $Ap(S, 30k+1) = \{a(30k+13) \mid a \in \{0,1,\ldots,15k\}\} \cup \{(30k+7)+a(30k+13) \mid a \in \{0,1,\ldots,15k-1\}\}$.*

Proof. Consider the equation $(15k+7)(30k+1) = (30k+7)+15k(30k+13)$ which is already given. Note that any element of $T = \{a(30k+13)|a \in \{0,1,\ldots,15k\}\} \cup \{(30k+7)+a(30k+13)|a \in \{0,1,\ldots,15k-1\}\}$ is also an element of $\mathrm{Ap}(S, 30k+1)$ by Lemma 3.1. Then the desired result follows from the fact that both T and $\mathrm{Ap}(S, 30k+1)$ have the same number of elements, namely, $2 \times (15k+1) - 1 = 30k+1$. □

As an immediate consequence of Theorem 3.2, we obtain the Frobenius number, pseudo-Frobenius number, genus, and type of the numerical semigroup $\langle 30k+1, 30k+7, 30k+13\rangle$.

Corollary 3.3. *Let $k \in \mathbb{N}$ and $S = \langle 30k+1, 30k+7, 30k+13 \rangle$. Then we have:*

(a) $F(S) = 450k^2 + 165k - 1$.

(b) $g(S) = 225k^2 + 90k$.

(c) $PF(S) = \{450k^2 + 165k - 1, 450k^2 + 165k - 7\}$ *and* $t(S) = 2$.

Proof. (a) This can be obtained by doing a simple computation as follows:

$$\begin{aligned} F(S) &= \max(\mathrm{AP}(S, 30k+1)) - (30k+1) \\ &= 15k(30k+13) - (30k+1) = 450k^2 + 165k - 1. \end{aligned}$$

(b) This also follows from the following direct computation:

$$\begin{aligned} g(S) = &\frac{1}{30k+1}\left(\frac{(15k)(15k+1)(30k+13)}{2} + (15k)(30k+7)\right. \\ &\left. + \frac{(15k-1)(15k)(30k+13)}{2}\right) - \frac{30k}{2} \\ = &\ 225k^2 + 90k. \end{aligned}$$

(c) It is clear by considering the maximal elements in the Apéry set of S as follows:

$$\begin{aligned} &\mathrm{maximals}_{\leq S}(\mathrm{AP}(S, 30k+1)) \\ &\qquad = \{15k(30k+13), (30k+7) + (15k-1)(30k+13)\}. \end{aligned}$$

This completes the proof. □

Similar arguments apply for other forms, and we summarize the analogous results in the next three subsequent subsections.

3.2 The second form: $S = \langle 30k+7, 30k+13, 30k+19 \rangle$

Let us record the following equations.

1. $(15k+10)(30k+7) = (30k+13) + (15k+3)(30k+19)$
2. $2(30k+13) = (30k+7) + (30k+19)$
3. $(15k+4)(30k+19) = (15k+9)(30k+7) + (30k+13)$

Using the above equations, we can describe the shape of a subset of the Apéry set $\mathrm{Ap}(S, 30k+7)$.

Lemma 3.4. *Let* $k \in \mathbb{N}$ *and* $S = \langle 30k+7, 30k+13, 30k+19\rangle$. *Then we have* $\{a(30k+19) \mid a \in \{0,1,\ldots,15k+3\}\} \cup \{(30k+13)+a(30k+19) \mid a \in \{0,1,\ldots,15k+2\}\} \subseteq Ap(S, 30k+7)$.

Now we are ready to suggest the following theorem for Apéry set of the sexy prime triplets of the form $(30k+7, 30k+13, 30k+19)$.

Theorem 3.5. *Let* $k \in \mathbb{N}$ *and* $S = \langle 30k+7, 30k+13, 30k+19\rangle$. *Then we have* $Ap(S, 30k+7) = \{a(30k+19) \mid a \in \{0,1,\ldots,15k+3\}\} \cup \{(30k+13)+a(30k+19) \mid a \in \{0,1,\ldots,15k+2\}\}$.

As before, we obtain the Frobenius number, pseudo-Frobenius number, genus, and type of $\langle 30k+7, 30k+13, 30k+19\rangle$ using Theorem 3.5.

Corollary 3.6. *Let* $k \in \mathbb{N}$ *and* $S = \langle 30k+7, 30k+13, 30k+19\rangle$. *Then we have:*

(a) $F(S) = 450k^2 + 345k + 50$.

(b) $g(S) = 225k^2 + 195k + 30$.

(c) $PF(S) = \{450k^2 + 345k + 50, 450k^2 + 345k + 44\}$ *and* $t(S) = 2$.

3.3 The third form: $S = \langle 30k+11, 30k+17, 30k+23\rangle$

Let us consider the following equations.

1. $(15k+12)(30k+11) = (30k+17) + (15k+5)(30k+23)$
2. $2(30k+17) = (30k+11) + (30k+23)$
3. $(15k+6)(30k+23) = (15k+11)(30k+11) + (30k+17)$

Using these equations, we also obtain the shape of a subset of the Apéry set $\mathrm{Ap}(S, 30k+11)$.

Lemma 3.7. *Let* $k \in \mathbb{N}$ *and* $S = \langle 30k+11, 30k+17, 30k+23\rangle$. *Then we have* $\{a(30k+23) \mid a \in \{0,1,\ldots,15k+5\}\} \cup \{(30k+17)+a(30k+23) \mid a \in \{0,1,\ldots,15k+4\}\} \subseteq Ap(S, 30k+11)$.

Thus we suggest the following theorem for Apéry set of the sexy prime triplets of the form $(30k+11, 30k+17, 30k+23)$.

Theorem 3.8. *Let* $k \in \mathbb{N}$ *and* $S = \langle 30k+11, 30k+17, 30k+23\rangle$. *Then we have* $Ap(S, 30k+11) = \{a(30k+23) | a \in \{0, 1, \dots, 15k+5\}\} \cup \{(30k+17) + a(30k+23) | a \in \{0, 1, \dots, 15k+4\}\}$.

We also get the Frobenius number, pseudo-Frobenius number, genus, and type of $\langle 30k+11, 30k+17, 30k+23\rangle$ using Theorem 3.8.

Corollary 3.9. *Let* $k \in \mathbb{N}$ *and* $S = \langle 30k+11, 30k+17, 30k+23\rangle$. *Then we have:*

(a) $F(S) = 450k^2 + 465k + 104$.

(b) $g(S) = 225k^2 + 255k + 60..$

(c) $PF(S) = \{450k^2 + 465k + 104, 450k^2 + 465k + 98\}$ *and* $t(S) = 2$.

3.4 The fourth form: $S = \langle 30k+17, 30k+23, 30k+29\rangle$

Finally, let us see the following equations.

1. $(15k+15)(30k+17) = (30k+23) + (15k+8)(30k+29)$
2. $2(30k+23) = (30k+17) + (30k+29)$
3. $(15k+9)(30k+29) = (15k+14)(30k+17) + (30k+23)$

Using the above equations we have the following lemma which indicates the shape of a subset of the Apéry set $\mathrm{Ap}(S, 30k+17)$.

Lemma 3.10. *Let* $k \in \mathbb{N}$ *and* $S = \langle 30k+17, 30k+23, 30k+29\rangle$. *Then we have* $\{a(30k+29) \mid a \in \{0, 1, \dots, 15k+8\}\} \cup \{(30k+23) + a(30k+29) \mid a \in \{0, 1, \dots, 15k+7\}\} \subseteq Ap(S, 30k+17)$.

The following theorem is to describe the Apéry set of the sexy prime triplets of the type $(30k+17, 30k+23, 30k+29)$.

Theorem 3.11. *Let* $k \in \mathbb{N}$ *and* $S = \langle 30k+17, 30k+23, 30k+29\rangle$. *Then we have* $Ap(S, 30k+17) = \{a(30k+29) | a \in \{0, 1, \dots, 15k+8\}\} \cup \{(30k+23) + a(30k+29) | a \in \{0, 1, \dots, 15k+7\}\}$.

As a result, we obtain the Frobenius number, pseudo-Frobenius number, genus, and type of $\langle 30k+17, 30k+23, 30k+29\rangle$ using Theorem 3.11.

Corollary 3.12. *Let* $k \in \mathbb{N}$ *and* $S = \langle 30k+17, 30k+23, 30k+29\rangle$. *Then we have:*

(a) $F(S) = 450k^2 + 645k + 215.$

(b) $g(S) = 225k^2 + 345k + 120..$

(c) $PF(S) = \{450k^2 + 645k + 215, 450k^2 + 645k + 209\}$ *and* $t(S) = 2.$

Acknowledgment. This work was supported by the research grant of Jeju National University in 2022.

References

[1] I.M. Aliev, P.M. Gruber, "Bounds on the number of numerical semigroups of a given genus", *J. Pure. Appl. Algebra,* **213,** (2009), 997–1001.

[2] D. Beihoffer, J., Hendry, A., Nijenhuis, S. Wagon, "Faster algorithms for Frobenius numbers", *Electron. J. Comb.,* **12,** (2005).

[3] S. Bocker, Z. and Lipták, "A fast and simple algorithm for the money changing problem", *Algorithmica,* **48,** (2007), 413–432.

[4] F. Curtis, "On formulas for the Frobenius number of a numerical semigroup", *Math. Scand.,* **67,** (1990), 190–192.

[5] L.E. Dickson, "A new extension of Dirichlet's theorem on prime numbers", *Messenger of Math.,* **33,** (1904), 155–161.

[6] B.K. Gil, J.W. Han, T.H. Kim, R.H. Koo, B.W. Lee, J.H. Lee, K.S. Nam, H.W. Park, P.S. Park, "Frobenius numbers of Pythagorean triples", *Int. J. Number Theory* **11,** (2015), 613–619.

[7] Z. Gu, X. Tang, "The Frobenius problem for a class of numerical semigroups", *Int. J. Number Theory,* **13,** (2017), 1335–1347.

[8] G.H. Hardy, J.E. Littlewood, "Some problems of 'partitio numerorum'; III: on the expression of a number as a sum of primes", *Acta Math.,* **44,** (1923), 1–70.

[9] G.H. Hardy, E.M. Wright, *An Introduction to the Theory of Numbers*, Oxford Univ. Press, 1979.

[10] B.R. Heap, M.S. Lynn, "On a linear Diophantine problem of Frobenius: an improved algorithm", *Numer. Math.,* **7,** (1965), 226–231.

[11] M. Hujter, B. Vizvari, "The exact solution to the Frobenius problem with three variables", *J. Ramanujan. Math. Soc.,* **2,** (1987), 117–143.

[12] M. Lepilov, J. O'Rourke, I. Swanson, "Frobenius numbers of numerical semigroups generated by three consecutive squares or cubes", *Semigroup Forum,* **91,** (2015), 238–259.

[13] G. Márquez-Campos, I. Ojeda, J.M. Tornero, "On the computation of the Apéry set of numerical monoids and affine semigroups", *Semigroup Forum,* **91,** (2015), 139–158.

[14] J.M. Marín, J.L. Ramírez Alfonsín, M.P. Revuelta, "On the Frobenius number of Fibonacci numerical semigroups", *Integers,* **7,** (2007), A14.

[15] D.C. Ong, V. Ponomarenko, "The Frobenius number of geometric sequences", *Integers,* **8,** (2008), A33.

[16] R.W. Owens, "An algorithm to solve the Frobenius problem", *Mathematics Magazine,*, **76,** (2003), 264–275.

[17] M. Raczunas, P. Chrzastowski-Wachtel, "A diophantine problem of Frobenius in terms of the least common multiple," *Discrete Math.,* **150,** (1996), 347–357.

[18] J.L. Ramírez-Alfonsín, "Complexity of the Frobenius problem", *Combinatorica,* **16,** (1996), 143–147.

[19] J.L. Ramírez-Alfonsín, O.J. Rødseth, "Numerical semigroups: Apéry sets and Hilbert series", *Semigroup Forum,* **79,** (2009), 323–340.

[20] A.M. Robles-Pérez, J.C. Rosales, "The Frobenius problem for numerical semigroups with embedding dimension equal to three", *Math. Comput.,* **81,** (2012), 1609–1617.

[21] A.M. Robles-Pérez, J.C. Rosales, "The Frobenius problem for prime k-tuplets", arXiv preprint, arXiv:2112.05501, (2021).

[22] O.J. Rødseth, "On a linear Diophantine problem of Frobenius", *J. Reine Angew. Math.,* **301,** (1978), 171–178.

[23] J. C. Rosales, Numerical semigroups with Apéry sets of unique expression, *J. Algebra,* **226,** (2000), 479–487.

[24] J.C. Rosales, M.B. Branco, D. Torrão, "The Frobenius problem for Thabit numerical semigroups", *J. Number Theory,* **155,** (2015), 85–99.

[25] J.C. Rosales, M.B. Branco, D. Torrão, "The Frobenius problem for repunit numerical semigroups", *Ramanujan J.,* **40,** (2016), 323–334.

[26] J.C. Rosales, M.B. Branco, D. Torrão, "The Frobenius problem for Mersenne numerical semigroups", *Math. Z.,* **286,** (2017), 1–9.

[27] J.C. Rosales, P.A. García-Sánchez, *Numerical Semigroups*, Springer Science & Business Media, New York, 2009.

[28] J.C. Rosales, P.A. García-Sánchez, J.I. García-García, J.J. Madrid, "Fundamental gaps in numerical semigroups", *J. Pure. Appl. Algebra,* **189,** (2004), 301–313.

[29] E.S. Selmer, "On the linear Diophantine problem of Frobenius", *J. Reine. Angew. Math.,* **293,** (1977), 1–17.

[30] K.H. Song, "The Frobenius problem for numerical semigroups generated by the Thabit numbers of the first, second kind base b and the Cunningham numbers", *Bull. Korean Math. Soc.,* **57,**no. 3, (2020), 623–647.

[31] K.H. Song, "The Frobenius problem for extended Thabit numerical semigroups", *Integers,* **21,** (2021), #A17.

[32] J.J. Sylvester, "Problem 7382", *The Educational Times and Journal of the College Of Preceptors, New Series,* **36,** (1883), 177.

[33] A. Tripathi, "Formulae for the Frobenius number in three variables", *J. Number Theory,* **170,** (2017), 368–389.