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The Complete Set of Non-negative Integer Solutions for the Diophantine Equation $(pq)^{2x} + p^y = z^2$, where p, q, x, y, z are non-negative integers with p prime and $p \nmid q$

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Abstract

In this article, we give the complete set of non-negative integer solutions for the Diophantine equation $(pq)^{2x}+p^y = z^2$, where p, q, x, y, z are non-negative integers with p prime and $p \nmid q$.

1 Introduction

Many researchers have recently studied the Diophantine equation of type $p^x + q^y = z^2$, where p, q are primes and x, y, z are non-negative integers (see,

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for instance, [1], [2], [3], [4], [5], [7], [8], [9], [10]). In this article, we give the complete set of non-negative integer solutions for the Diophantine equation

$$(pq)^{2x} + p^y = z^2 \tag{1.1}$$

where p, q, x, y, z are non-negative integers with p prime and $p \nmid q$. From (1.1), we have $p^y = (z - (pq)^x)(z + (pq)^x)$. Since p is prime, there exists a non-negative integer u such that

$$z - (pq)^x = p^u \tag{1.2}$$

and

$$z + (pq)^x = p^{y-u}. \tag{1.3}$$

Since (1.2), (1.3) and $z - (pq)^x < z + (pq)^x$, we get $y > 2u$ and

$$2(pq)^x = p^u \left(p^{y-2u} - 1\right). \tag{1.4}$$

2 Main results

We start this section with the following famous and helpful theorem:

Theorem 2.1. *[6] (Mihăilescu's Theorem)* $(a, b, x, y) = (3, 2, 2, 3)$ *is the only one non-negative integer solution of the Diophantine equation* $a^x - b^y = 1$*, where* a, b, x, y *are integers with* $\min\{a, b, x, y\} > 1$.

In the first part, we focus on finding all non-negative integer solutions of (1.1) when $p = 2$.

Theorem 2.2. *Let* p, q, x, y, z *be non-negative integers with* p *prime and* $p \nmid q$*. If* $p = 2$*, then the Diophantine equation (1.1) has non-negative integer solutions*
$(p, q, x, y, z) \in \{(2, q, 0, 3, 3), (2, 1, x, 2x + 3, 3 \cdot 2^x), (2, 2^{y-4} - 1, 1, y, 2^{y-3} + 2)\}$.

Proof. Since $p = 2$, by (1.4) we see that $2^{x+1}q^x = 2^u (2^{y-2u} - 1)$. Since $2 \nmid q$, we get $u = x + 1$ and so

$$2^{y-2u} - q^x = 1. \tag{2.5}$$

Case 1. $x = 0$. Then $u = 1$. From (2.5), we obtain $2^{y-2} = 2$. This implies that $y = 3$. From (1.2), we have $z = 3$ and so $(p, q, x, y, z) = (2, q, 0, 3, 3)$.
Case 2. $x = 1$. Then $u = 2$. From (2.5) and (1.2), we get $q = 2^{y-4} - 1$ and $z = 2^{y-3} + 2$, respectively. Thus $(p, q, x, y, z) = (2, 2^{y-4} - 1, 1, y, 2^{y-3} + 2)$.

Case 3. $x > 1$. Since $2 \nmid q$, $q \geq 1$. We consider the following cases:

Subcase 3.1: $q = 1$. From (2.5), we get $y - 2u = 1$ and so $y = 2x + 3$. From (1.2), we have $z = 3 \cdot 2^x$. Then $(p, q, x, y, z) = (2, 1, x, 2x + 3, 3 \cdot 2^x)$.

Subcase 3.2: $q > 1$. Assuming that $y - 2u = 1$, from (2.5), we get $x = 0$, a contradiction. Thus $y - 2u > 1$ and so $\min\{2, q, y - 2u, x\} > 1$. This is impossible by (2.5) and Theorem 2.1. □

Corollary 2.3. *Let p, q, x, y, z be non-negative integers with p and q distinct primes. If $p = 2$ and y is even, then the Diophantine equation (1.1) has only one non-negative integer solution $(p, q, x, y, z) = (2, 3, 1, 6, 10)$.*

Proof. Since y is even and using Theorem 2.2, $q = 2^{y-4} - 1$, $x = 1$ and $z = 2^{y-3} + 2$. Set $y = 2k$, for some non-negative integer k. Then $q = 2^{2k-4} - 1 = \left(2^{k-2} - 1\right)\left(2^{k-2} + 1\right)$. Since q is prime, $2^{k-2} - 1 = 1$ and $2^{k-2} + 1 = q$. Thus $k = 3$ and $q = 3$. Therefore, $y = 6$ and $z = 10$. Consequently, $(p, q, x, y, z) = (2, 3, 1, 6, 10)$. □

In the second part, we investigate all non-negative integer solutions of (1.1) when $q = 2$.

Theorem 2.4. *Let p, q, x, y, z be non-negative integers with p prime and $p \nmid q$. If $q = 2$, then the Diophantine equation (1.1) has non-negative integer solutions*
$(p, q, x, y, z) \in \{(3, 2, 2, 6, 45), \left(1 + 2^{x+1}, 2, x, 1 + 2x, (1 + 2^{x+1})^x (1 + 2^x)\right)\}$.

Proof. Since $q = 2$, by (1.4) we see that $2^{x+1}p^x = p^u\left(p^{y-2u} - 1\right)$. Since p is prime and $p \nmid 2$, we have $u = x$ and so

$$p^{y-2u} - 2^{x+1} = 1. \tag{2.6}$$

Case 1. $x = 0$. Then $u = 0$. From (2.6), we get $p^y = 3$. This implies $p = 3$ and $y = 1$. From (1.2), we have $z = 2$. Thus $(p, q, x, y, z) = (3, 2, 0, 1, 2)$.

Case 2. $x > 0$. We consider the following two subcases:

Subcase 2.1: $y - 2u = 1$. Then $y = 1 + 2x$. From (2.6) and (1.2), we have $p = 1 + 2^{x+1}$ and $z = (1 + 2^{x+1})^x (1 + 2^x)$, respectively. This implies that $(p, q, x, y, z) = \left(1 + 2^{x+1}, 2, x, 1 + 2x, (1 + 2^{x+1})^x (1 + 2^x)\right)$.

Subcase 2.2: $y - 2u > 1$. Then $\min\{p, 2, y - 2u, x + 1\} > 1$. By Theorem 2.1, we get $p = 3, y - 2u = 2$ and $x + 1 = 3$. Thus $x = 2$ and $y = 6$. From (1.2), we have $z = 45$. Then $(p, q, x, y, z) = (3, 2, 2, 6, 45)$. □

By Theorem 2.4, we have the following corollary:

Corollary 2.5. *Let p,q,x,y,z be non-negative integers with p and q distinct primes. If $q=2$ and y is even, then the Diophantine equation (1.1) has only one non-negative integer solution* $(p,q,x,y,z)=(3,2,2,6,45)$.

In the third part, we find out all non-negative integer solutions of (1.1) when $p\neq 2$ and $q\neq 2$.

Theorem 2.6. *Let p,q,x,y,z be non-negative integers with p prime and $p\nmid q$. If $p\neq 2$ and $q\neq 2$, then the Diophantine equation (1.1) has non-negative integer solutions*
$(p,q,x,y,z)\in\{(3,q,0,1,2)\}\cup$
$\left\{\left(p,\sqrt[x]{\frac{p^{y-2x}-1}{2}},x,y,p^x+p^x\left(\frac{p^{y-2x}-1}{2}\right)\right):\sqrt[x]{\frac{p^{y-2x}-1}{2}}\in\mathbb{N}\right\}$.

Proof. Since p is prime, $p\nmid q$ with $p\neq 2$ and $q\neq 2$, it follows from (1.4) that $u=x$ and so

$$p^{y-2u}-2q^x=1. \tag{2.7}$$

Case 1. $x=0$. Then $u=0$. From (2.7), we get $p^y=3$. This implies that $p=3$ and $y=1$. From (1.2), we have $z=2$. Then $(p,q,x,y,z)=(3,q,0,1,2)$.
Case 2. $x>0$. From (2.7) and (1.2), $q^x=\frac{p^{y-2x}-1}{2}$ and $z=p^x+p^x\left(\frac{p^{y-2x}-1}{2}\right)$, respectively. Then $(p,q,x,y,z)=\left(p,\sqrt[x]{\frac{p^{y-2x}-1}{2}},x,y,p^x+p^x\left(\frac{p^{y-2x}-1}{2}\right)\right)$. □

Remark 2.7. *By Theorem 2.6, we see that* $(p,q,x,y,z)=(3,11,2,9,1098)$ *is a non-negative integer solution of (1.1).*

Corollary 2.8. *Let p,q,x,y,z be non-negative integers with p and q distinct odd primes. If y is even, then the Diophantine equation (1.1) has no non-negative integer solution.*

Proof. Since y is even, by Theorem 2.6, we see that $q^x=\frac{p^{y-2x}-1}{2}$. Set $y=2k$, for some non-negative integer k. Then $q^x=\frac{p^{2k-2x}-1}{2}=\frac{(p^{k-x}-1)(p^{k-x}+1)}{2}$. Since p is odd, $p^{k-x}-1$ and $p^{k-x}+1$ are even and so $2\mid q$, a contradiction. □

By Corollary 2.3, 2.5 and 2.8, we have the following theorem:

Theorem 2.9. *Let p,q,x,y,z be non-negative integers with p and q distinct primes. If y is even, then the Diophantine equation (1.1) has exactly two non-negative integer solutions* $(p,q,x,y,z)\in\{(2,3,1,6,10),(3,2,2,6,45)\}$.

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