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On the Diophantine equation $(pq)^x + (pq)^{2s}\, n^y = z^2$, where p and q are prime numbers

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Abstract

In this paper, we study the Diophantine equation $(pq)^x+(pq)^{2s}n^y = z^2$, where p and q are prime numbers such that $pq \equiv 3 \pmod{20}$ and $n \equiv 5 \pmod{20}$. We show that a non-negative integer solution of such equation exists only if $pq + 1$ is a square. The solution is also in the form $(x, y, z) = (1 + 2s, 0, (pq)^s\sqrt{pq + 1})$.

1 Introduction

There are a lot of studies on the Diophantine equation of type

$$a^x + b^y = z^2.$$

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In 2011, Suvarnamani [1] showed that $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no non-negative integer solution. In 2012, Sroysang [2] showed that $31^x + 32^y = z^2$, has no non-negative integer solution. In 2013, Sroysang [3] showed that the $5^x + 13^y = z^2$ has no non-negative integer solution. In 2014, Sarasit and Chotchaisthit [4] found that $(x, y, z) = (1+2s, 0, 2(3^s))$ is a solution to $3^x+3^{2s}n^y = z^{2t}$, where $t = 1$ and n, s are non-negative integers. In 2015, Sarasit and Chotchaisthit [5] found that $(x, y, z) = (1+2s, 0, 18(323^s))$ is a solution to $323^x + 323^{2s}n^y = z^{2t}$, where $t = 1$ and n, s are non-negative integers. In 2020, Orosram and Comemuang [6] showed that $(x, y, z) = (1, 0, 3)$ is the unique non-negative integer solution to $8^x + n^y = z^2$. In 2021, Tangjai and Chubthaisong [7] studied the Diophantine equation $3^x+p^y = z^2$. In 2022, Orosram, Jaidee and Tangjai [8] studied the Diophantine equation $(p + 2)^x + (2p + 1)^y = z^2$. N. Viriyapong and C. Viriyapong [9] studied the Diophantine equation $n^x + p^y = z^2$, where $n \equiv 2 \pmod{57}$.

In this paper, we find all non-negative integer solutions of the Diophantine equation $(pq)^x+(pq)^{2s}n^y = z^2$, when p, q are prime numbers and s, n are non-negative integer such that $pq \equiv 3 \pmod{20}$ and $n \equiv 5 \pmod{20}$. We use the method of proof appearing in [4, 5] to derive the result.

2 Main results

In this paper, we assume that n is a non-negative integer and p, q are prime numbers.

Proposition 2.1. *(Catalans conjecture) The Diophantine equation $a^x-b^y = 1$, has the unique solution $(a, b, x, y) = (3, 2, 2, 3)$, where a, b, x and y are integers such that $\min\{a, b, x, y\} > 1$.*

Lemma 2.2. *Let p and q be prime numbers such that $pq \equiv 3 \pmod{20}$. A non-negative integer solution to $(pq)^x+1 = z^2$ exists only if $pq+1$ is a square. In addition, the solutions are $(x, z) = (1, \sqrt{pq + 1})$.*

Proof. Let (x, z) be a non-negative integer solution of $(pq)^x+1 = z^2$. If $x > 1$, then $z > 1$. By Catalan's conjecture there is no non-negative integer solution. If $x = 0$, then $z^2 = 2$, which is impossible. If $x = 1$, then $z = \sqrt{pq + 1}$. Thus, the solution exists if and only if $pq + 1$ is a square. We also have $(x, z) = (1, \sqrt{pq + 1})$. □

Lemma 2.3. *Let $pq \equiv 3 \pmod{20}$. If x is an odd number, then $(pq)^x \equiv 2 \pmod{5}$ or $(pq)^x \equiv 3 \pmod{5}$*

Proof. Let x be an odd number. There exists an integer $k \geq 0$ such that $x = 2k+1$. Since $3^2 \equiv -1 \pmod 5$, it follows that $3^{2k+1} \equiv 3(-1)^k \pmod 5$. Hence

$$(pq)^x \equiv 3^{2k+1} \equiv \begin{cases} 2 \pmod 5 & \text{if } k \text{ is odd,} \\ 3 \pmod 5 & \text{if } k \text{ is even.} \end{cases}$$

□

Theorem 2.4. *For a non-negative integer n such that $n \equiv 5 \pmod{20}$, let p, q be prime numbers such that $pq \equiv 3 \pmod{20}$. A non-negative integer solution to $(pq)^x + n^y = z^2$ exists only if $pq + 1$ is a square. In addition, the solution are $(x, y, z) = (1, 0, \sqrt{pq+1})$.*

Proof. Let (x, y, z) be a non-negative integer solution of $(pq)^x + n^y = z^2$. We consider the following cases.
Case 1. $y = 0$. By Lemma 2.2, we have $(x, y, z) = (1, 0, \sqrt{pq+1})$, where $\sqrt{pq+1}$ is a non-negative integer.
Case 2. $y \geq 1$. Suppose that there exists a non-negative integer solution (x, y, z) for the Diophantine equation $(pq)^x + n^y = z^2$. Since $n \equiv 5 \pmod{20}$ and $pq \equiv 3 \pmod{20}$, it follows that $n \equiv 1 \pmod 4$ and $pq \equiv 3 \pmod 4$. Since z is an even number, we have $z^2 \equiv 0 \pmod 4$. Thus $(pq)^x + n^y \equiv 0 \pmod 4$. Since $n^y \equiv 1 \pmod 4$, it follows that $(pq)^x \equiv -1 \pmod 4$. This implies that x is an odd number. By Lemma 2.3 we have $(pq)^x \equiv 2 \pmod 5$ or $(pq)^x \equiv 3 \pmod 5$. Since $n^y \equiv 0 \pmod 5$, we have $z^2 = (pq)^x + n^y \equiv 2, 3 \pmod 5$ but $z^2 \equiv 0, 1, 4 \pmod 5$ which is a contradiction. Thus, in this case, there is no non-negative integer solution. □

Example 2.5. $(x, y, z) = (1, 0, 12)$ *is the unique non-negative integer solution (x, y, z) for the Diophantine equation $143^x + 5^y = z^2$.*

Lemma 2.6. *Let A, n be non-negative integer with $A \equiv 1 \pmod 4$ and $n \equiv 5 \pmod{20}$. The Diophantine equation $1 + An^y = z^2$ has no non-negative integer solution (y, z).*

Proof. Suppose that there exist non-negative integers y, z satisfying the equation $1 + An^y = z^2$. Since $n \equiv 5 \pmod{20}$, it follows that $n \equiv 1 \pmod 4$. So $z^2 = 1 + An^y \equiv 2 \pmod 4$ contradicts $z^2 \equiv 0, 1 \pmod 4$. Therefore, the considered equation has no non-negative integer solution (y, z). □

Lemma 2.7. *Let p and q be prime numbers and m be a non-negative integer such that $m \geq 2$ and $p \neq q$. A Diophantine equation $pq + (pq)^m n^y = z^2$ has no non-negative integer solution (y, z).*

Proof. Suppose that there exist non-negative integers y, z such that $pq + (pq)^m n^y = z^2$. This implies that $pq \mid z^2$. Since p and q are prime numbers, it follows that $pq \mid z$. We write $z = (pq)r$, for some $r \in \mathbb{Z}^+$. By substituting $z = (pq)r$, we have $pq + (pq)^m n^y = (pq)^2 r^2$. Thus $1 + (pq)^{m-1} n^y = (pq) r^2$. Hence $1 = pq\,(r^2 - (pq)^{m-2} n^y)$. So $pq \mid 1$, a contradiction. Therefore, $pq + (pq)^m n^y = z^2$ has no non-negative integer solution. □

Theorem 2.8. *For non-negative integers s and n where $n \equiv 5 \pmod{20}$, let p, q be prime numbers such that $pq \equiv 3 \pmod{20}$ and $p \neq q$. The solution to $(pq)^x + (pq)^{2s} n^y = z^2$ exists only if $pq + 1$ is a square. In addition, the solutions are $(x, y, z) = (1 + 2s, 0, (pq)^s \sqrt{pq + 1})$.*

Proof. We use mathematical induction on s. If $s = 0$, then the only non-negative integer solution to $(pq)^x + n^y = z^2$ is $(x, y, z) = (1, 0, \sqrt{pq + 1})$ by Theorem 2.4. For a non-negative integer k, suppose that $(pq)^x + (pq)^{2k} n^y = z^2$ has a unique non-negative integer solution $(x, y, z) = (1+2k, 0, (pq)^k \sqrt{pq + 1})$. We consider the following cases.

Case 1. $x = 0$. Since $(pq)^{2(k+1)} \equiv 1 \pmod 4$ and by Lemma 2.6, the Diophantine equation $1 + (pq)^{2(k+1)} n^y = z^2$ has no non-negative integer solution.

Case 2. $x = 1$. By lemma 2.7, the Diophantine equation $pq + (pq)^{2(k+1)} n^y = z^2$ has no non-negative integer solution.

Case 3. $x \geq 2$. Note that $(pq)^x + (pq)^{2(k+1)} n^y = z^2$ can be written as $(pq)^{x-2} + (pq)^{2k} n^y = \left(\frac{z}{pq}\right)^2$. Since $x - 2$ is a non-negative integer, it follows that $(\frac{z}{pq})^2$ is also a non-negative integer and so is $\frac{z}{pq}$. Let $u = x - 2$ and $v = \frac{z}{pq}$. By the assumption, we have that $(pq)^u + (pq)^{2k} n^y = v^2$ has the non-negative integer solutions $(u, y, v) = (1 + 2k, 0, (pq)^k \sqrt{pq + 1})$. Thus $(x, y, z) = (1 + 2(k + 1), 0, (pq)^{k+1} \sqrt{pq + 1})$ are non-negative integer solution to $(pq)^x + (pq)^{2(k+1)} n^y = z^2$. Therefore, $(pq)^x + (pq)^{2s} n^y = z^2$ has the non-negative integer solutions $(x, y, z) = (1 + 2s, 0, (pq)^s \sqrt{pq + 1})$. □

Example 2.9. $(x, y, z) = (1 + 2s, 0, 12(143)^s)$ *are the non-negative integer solutions (x, y, z) to $143^x + (143)^{2s} n^y = z^2$.*

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