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Co-cobalancing numbers and Balancing numbers

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Abstract

A positive integer n is called a co-cobalancing number if n is a solution of the equation $1+2+3+\cdots+(n+1)=(n+1)+(n+2)+\cdots+(n+r)$ for some positive integer r. Our purpose in this paper is to present a function of co-cobalancing numbers and recurrence relations for co-cobalancing numbers, some relations among balancing numbers and co-cobalancing numbers, and some interesting results on co-cobalancing numbers by using Binet's formula. Moreover, we provide an application of co-cobalancing numbers to a Diophantine equation.

1 Introduction

The definition of balancing numbers was introduced in 1999 by Behera and Panda [1]:
a positive integer n is called a balancing number if n is a solution of

$$1+2+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r), \tag{1.1}$$

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for some positive integer r.

r is called the balancer corresponding to the balancing number n. They also established many important results on balancing numbers. For any balancing number x, $F(x) = 3x + \sqrt{8x^2+1}$ is a balancing number.

Let B_n be the n^{th} balancing numbers. Set $B_0 = 1, B_1 = 6, B_2 = 35$ and so on. The second order linear recurrence:

$$B_{n+1} = 6B_n - B_{n-1}; \quad n = 1, 2, 3, \cdots .$$

The non-linear first order recurrence:

$$B_{n+1} = 3B_n + \sqrt{8B_n^2+1}; \quad n = 0, 1, 2, \cdots .$$

and

$$B_{n-1} = 3B_n - \sqrt{8B_n^2+1}; \quad n = 1, 2, 3, \cdots .$$

The Binet form:

$$B_n = \frac{\lambda^{n+1} - \beta^{n+1}}{\lambda - \beta}, \quad n = 0, 1, 2, \cdots$$

where $\lambda = 3 + \sqrt{8}$ and $\beta = 3 - \sqrt{8}$.

For $n = 1, 2, 3, \cdots$, R_n is the n^{th} balancer. Set $R_1 = 2, R_2 = 14, R_3 = 84$ and so on. Here are some results on balancing numbers and balancers:
For $n = 1, 2, 3, \cdots$

$$B_n = \frac{(2R_n + 1) + \sqrt{8R_n^2 + 8R_n + 1}}{2}.$$

and

$$R_n = \frac{-(2B_n + 1) + \sqrt{8B_n^2 + 1}}{2}.$$

Later in 2005, Panda and Ray [2] defined a cobalancing number $n \in I^+$ by

$$1 + 2 + \cdots + n = (n+1) + (n+2) + \cdots + (n+r), \tag{1.2}$$

for some positive integer r. r is called the cobalancer corresponding to the cobalancing number n.

We examine the notion of the co-cobalancing numbers and investigate their properties. Moreovers, we derive some relation among the balancing numbers and co-cobalancing numbers. Furthermore, we obtain some interesting results on co-cobalancing numbers by using Binet's formula.

2 Main results

Let n be a positive integer such that n is a solution of the equation

$$1 + 2 + \cdots + (n+1) = (n+1) + (n+2) + \cdots + (n+r), \tag{2.3}$$

for some positive integer r. n is called a co-cobalancing number and r the co-cobalancer corresponding to the co-cobalncing number n.
For example, $3, 15$ and 85 are co-cobalancers corresponding to co-cobalancing numbers $5, 34$ and 203, respectively. By (2.3), n is a co-cobalancing number with co-cobalancer r if and only if

$$(n+1)^2 = \frac{(n+r)(n+r+1)}{2} \tag{2.4}$$

and thus,

$$r = \frac{-(2n+1) + \sqrt{8n^2 + 16n + 9}}{2}. \tag{2.5}$$

By (2.4), we have n is a co-cobalancing number if and only if $(n+1)^2$ is a triangular number. In addition, by (2.5), n is a co-cobalancing number if and only if $8n^2 + 16n + 9$ is a perfect square.

2.1 Some functions that generate co-cobalancing numbers

For any co-cobalancing number x, consider the following functions:

$$\overline{f}(x) = (3x + 2) + \sqrt{8x^2 + 16x + 9}. \tag{2.6}$$

and

$$\overline{g}(x) = (17x + 16) + 6\sqrt{8x^2 + 16x + 9}. \tag{2.7}$$

First, we prove that the above functions always generate co-cobalancing numbers.

Theorem 2.1. *For any co-cobalancing number x, $\overline{f}(x)$ and $\overline{g}(x)$ are also co-cobalancing numbers.*

Proof. Let x be a co-cobalancing number. Suppose that $u = \overline{f}(x)$. It is obvious that $x < u$ and

$$x = (3u + 2) - \sqrt{8u^2 + 16u + 9}.$$

Since x and u are nonnegative integers, $8u^2+16u+9$ must be a perfect square. Hence u is a co-cobalancing number. Since $\overline{f}(\overline{f}(x)) = \overline{g}(x)$, it follows that $\overline{g}(x)$ is also a co-cobalancing number. This completes the proof of Theorem 2.1. □

Next, we show that for any co-cobalancing number x, $\overline{f}(x)$ is the co-cobalancing number next to x.

Theorem 2.2. *If x is any co-cobalancing number, then the co-cobalancing number next to x is $\overline{f}(x) = (3x+2) + \sqrt{8x^2+16x+9}$ and the previous one is $\overline{f}_1(x) = (3x+2) - \sqrt{8x^2+16x+9}$.*

Proof. The proof of $\overline{f}(x) = (3x+2) + \sqrt{8x^2+16x+9}$ is the co-cobalancing number next to x is similar to that of Theorem 3.1 in [1] and since $\overline{f}(\overline{f}_1(x)) = x$, we have $\overline{f}_1(x)$ is the largest co-cobalancing number less than x. □

2.2 Recurrence relations among co-cobalancing numbers and balancing numbers

For $n = 1, 2, \cdots$, let $\overline{B}_n$ be the nth co-cobalancing number. We define $\overline{B}_0 = 0$ (the reason is that $8 \cdot 0^2 + 16 \cdot 0 + 9 = 9$ is a perfect square). We know that $\overline{B}_1 = 5$, $\overline{B}_2 = 34$, $\overline{B}_3 = 203$ and so on. In section 2.1, we proved that if $\overline{B}_n$ is the nth co-cobalancing number, then

$$\begin{aligned} \overline{B}_{n+1} &= (3\overline{B}_n + 2) + \sqrt{8\overline{B}_n^2 + 16\overline{B}_n + 9} \qquad \text{and} \\ \overline{B}_{n-1} &= (3\overline{B}_n + 2) - \sqrt{8\overline{B}_n^2 + 16\overline{B}_n + 9}. \end{aligned} \tag{2.8}$$

It is clear that the co-cobalancing numbers obey the following recurrence relation:

$$\overline{B}_{n+1} = 6\overline{B}_n - \overline{B}_{n-1} + 4, n \geq 1. \tag{2.9}$$

From the recurrence relation for balancing numbers $B_{n+1} = 6B_n - B_{n-1}$ for $n \geq 1$ and the recurrence relation for co-cobalancing numbers $\overline{B}_{n+1} = 6\overline{B}_n - \overline{B}_{n-1} + 4$ for $n \geq 1$, we obtain some other interesting relations.

Theorem 2.3. *Let B_n be the n^{th} balancing number and $\overline{B}_n$ be the n^{th} co-cobalancing number for $n \geq 1$. Then*

(a) $(\overline{B}_n - 2)^2 = \overline{B}_{n+1}\overline{B}_{n-1} + 9$,

(b) *for* $n > k \geq 1$,

$$\overline{B}_n = B_k\overline{B}_{n-k} - B_{k-1}\overline{B}_{n-k-1} + (B_k - B_{k-1} - 1),$$

(c) $\overline{B}_{2n} = B_n(\overline{B}_n + 1) - B_{n-1}(\overline{B}_{n-1} + 1) - 1,$

(d) $\overline{B}_{2n+1} = B_{n+1}(\overline{B}_n + 1) - B_n(\overline{B}_{n-1} + 1) - 1,$

(e) $\overline{B}_n = B_n - 1.$

Proof. **(a)** From (2.8), we get

$$\begin{aligned}\overline{B}_{n+1}\overline{B}_{n-1} &= \left[(3\overline{B}_n + 2) + \sqrt{8\overline{B}_n^2 + 16\overline{B}_n + 9}\right]\left[(3\overline{B}_n + 2) - \sqrt{8\overline{B}_n^2 + 16\overline{B}_n + 9}\right] \\ &= (3\overline{B}_n + 2)^2 - (8\overline{B}_n^2 + 16\overline{B}_n + 9) \\ &= (\overline{B}_n - 2)^2 - 9.\end{aligned}$$

Hence, $(\overline{B}_n - 2)^2 = \overline{B}_{n+1}\overline{B}_{n-1} + 9$.

(b) Let n and k be positive integers such that $n > k \geq 1$. The proof of **(b)** is based on mathematical induction on k. Clearly, **(b)** is true for $n > 1$ and $k = 1$. Assume that **(b)** is true for $k = r$. That is, $\overline{B}_n = B_r\overline{B}_{n-r} - B_{r-1}\overline{B}_{n-r-1} + (B_r - B_{r-1} - 1)$. Thus

$$\begin{aligned}&B_{r+1}\overline{B}_{n-r-1} - B_r\overline{B}_{n-r-2} + B_{r+1} - B_r - 1 \\ &\quad= (6B_r - B_{r-1})\overline{B}_{n-r-1} - B_r \cdot \overline{B}_{n-r-2} + 6B_r - B_{r-1} - B_r - 1 \\ &\quad= B_r(6\overline{B}_{n-r-1} - \overline{B}_{n-r-2} + 4) - B_{r-1} \cdot \overline{B}_{n-r-1} + B_r - B_{r-1} - 1 \\ &\quad= B_r\overline{B}_{n-r} - B_{r-1} \cdot \overline{B}_{n-r-1} + B_r - B_{r-1} - 1 \\ &\quad= \overline{B}_n.\end{aligned}$$

Therefore, **(b)** is true for $k = r + 1$. This completes the proof of **(b)**.

(c) The proof of **(c)** follows by replacing n by $2n$ and k by n in **(b)**.

(d) Similarly, the proof of **(d)** follows by replacing n by $2n+1$ and k by $n+1$ in **(b)**.

(e) By **(b)**,

$$\overline{B}_n = B_k\overline{B}_{n-k} - B_{k-1}\overline{B}_{n-k-1} + (B_k - B_{k-1} - 1)$$

for $n > k \geq 1$. Thus, for $k = n - 1$, we have

$$\begin{aligned}\overline{B}_n &= B_{n-1}\overline{B}_1 - B_{n-2}\overline{B}_0 + (B_{n-1} - B_{n-2} - 1) \\ &= 6B_{n-1} - B_{n-2} - 1 = B_n - 1.\end{aligned}$$

This completes the proof of Theorem 2.3. □

For $n = 1, 2, 3, \cdots$, let $\overline{R}_n$ is the n^{th} co-cobalancer, set $\overline{R}_1 = 3, \overline{R}_2 = 15, \overline{R}_3 = 85$ and so on. We know that

$$\overline{B}_n = \frac{(2\overline{R}_n - 3) + \sqrt{8\overline{R}_n^2 - 8\overline{R}_n + 1}}{2}$$

and

$$\overline{R}_n = \frac{-(2\overline{B}_n + 1) + \sqrt{8\overline{B}_n^2 + 16\overline{B}_n + 9}}{2}.$$

2.3 Relations among co-cobalancing numbers and co-cobalancers

Theorem 2.4. *If n is a natural number, then*

(a) $\overline{B}_n - \overline{B}_{n-1} = 2\overline{R}_n - 1$,

(b) $2\overline{B}_n = 5\overline{R}_n - \overline{R}_{n-1} - 4$,

(c) $2\overline{B}_n = \overline{R}_{n+1} - \overline{R}_n - 2$.

Proof. (a) Since

$$\begin{aligned}\overline{B}_n - \overline{B}_{n-1} &= \overline{B}_n - \left[(3\overline{B}_n + 2) - \sqrt{8\overline{B}_n^2 + 16\overline{B}_n + 9}\right] \\ &= -2\overline{B}_n - 1 + \sqrt{8\overline{B}_n^2 + 16\overline{B}_n + 9} - 1 \\ &= 2\Big[\frac{-(2\overline{B}_n + 1) + \sqrt{8\overline{B}_n^2 + 16\overline{B}_n + 9}}{2}\Big] - 1 \\ &= 2\overline{R}_n - 1,\end{aligned}$$

hence $\overline{B}_n - \overline{B}_{n-1} = 2\overline{R}_n - 1$ from which (a) follows.

The proof of (b) follows by

$$\begin{aligned}
\overline{R}_n - \overline{R}_{n-1} &= \overline{R}_n - \left[(3\overline{R}_n - 1) - \sqrt{8\overline{R}_n^2 - 8\overline{R}_n + 1}\right] \\
&= \overline{R}_n - 3\overline{R}_n + 1 + \sqrt{8\overline{R}_n^2 - 8\overline{R}_n + 1} \\
&= -2\overline{R}_n + 1 + \sqrt{8\overline{R}_n^2 - 8\overline{R}_n + 1} \\
&= -4\overline{R}_n + 4 + 2\overline{R}_n - 3 + \sqrt{8\overline{R}_n^2 - 8\overline{R}_n + 1} \\
&= -4\overline{R}_n + 4 + 2\overline{B}_n.
\end{aligned}$$

$$= \overline{B}_n - 3\overline{B}_n - 2 + \sqrt{8\overline{B}_n^2 + 16\overline{B}_n + 9}.$$

Hence $2\overline{B}_n = 5\overline{R}_n - \overline{R}_{n-1} - 4$. This completes the proof of (b).

The proof of (c) follows by

$$\begin{aligned}
\overline{R}_n + \overline{R}_{n+1} &= \overline{R}_n + (3\overline{R}_n - 1) + \sqrt{8\overline{R}_n^2 - 8\overline{R}_n + 1} \\
&= 4\overline{R}_n - 1 + \sqrt{8\overline{R}_n^2 - 8\overline{R}_n + 1} \\
&= 2\overline{R}_n - 3 + \sqrt{8\overline{R}_n^2 - 8\overline{R}_n + 1} + 2\overline{R}_n + 2 \\
&= 2\overline{B}_n + 2\overline{R}_n + 2.
\end{aligned}$$

Hence $2\overline{B}_n = \overline{R}_{n+1} - \overline{R}_n - 2$. The proof of (c) is complete. □

Theorem 2.5. *If n is a natural number, then $R_n = \overline{R}_n - 1$.*

Proof. Since

$$\begin{aligned}
R_n &= \frac{-(2B_n + 1) + \sqrt{8B_n^2 + 1}}{2} \\
&= \frac{-(2(\overline{B}_n + 1) + 1) + \sqrt{8(\overline{B}_n + 1)^2 + 1}}{2} \\
&= \frac{-2\overline{B}_n - 3 + \sqrt{8\overline{B}_n^2 + 16\overline{B}_n + 9}}{2} \\
&= \frac{-2\overline{B}_n - 1 + \sqrt{8\overline{B}_n^2 + 16\overline{B}_n + 9} - 2}{2} \\
&= \overline{R}_n - 1.
\end{aligned}$$

Hence $R_n = \overline{R}_n - 1$. □

2.4 Binet form for co-cobalancing numbers

In the previous section, we obtained the recurrence relation

$$\overline{B}_{n+1} - 6\overline{B}_n + \overline{B}_{n-1} - 4 = 0 \text{ for } n \geq 1,$$

which is a second-order linear nonhomogeneous difference equation with constant coefficients. Let $\overline{C}_n = \overline{B}_n + 1$ for $n \geq 0$. Hence $\overline{C}_{n+1} = 6\overline{C}_n - \overline{C}_{n-1}$, which is homogeneous. The general solution of this equation is

$$\overline{C}_n = A\lambda^n + B\beta^n \tag{2.10}$$

where $\lambda = 3 + \sqrt{8}$ and $\beta = 3 - \sqrt{8}$ are roots of the characteristic equation

$$x^2 - 6x + 1 = 0.$$

Substituting $\overline{C}_0 = 1$ and $\overline{C}_1 = 6$ into (2.10), we get

$$1 = A + B$$

$$6 = A\lambda + B\beta.$$

We obtain

$$A = \frac{\lambda}{\lambda - \beta} \text{ and } B = -\frac{\beta}{\lambda - \beta}.$$

Thus,

$$\overline{C}_n = \frac{\lambda^{n+1} - \beta^{n+1}}{\lambda - \beta}, n = 0, 1, 2, \cdots$$

which implies that

$$\overline{B}_n = \frac{\lambda^{n+1} - \beta^{n+1}}{\lambda - \beta} - 1, n = 0, 1, 2, \cdots .$$

The above discussion proves the following theorem:

Theorem 2.6. *If $\overline{B}_n$ is the nth co-cobalancing number, then its Binet form is*

$$\overline{B}_n = \frac{\lambda^{n+1} - \beta^{n+1}}{\lambda - \beta} - 1, n = 0, 1, 2, \cdots ,$$

where $\lambda = 3 + \sqrt{8}$ and $\beta = 3 - \sqrt{8}$

Theorem 2.7. *If m and n are natural numbers and $m > n$, then*

$$(\overline{B}_{m+n} + 1)(\overline{B}_{m-n} + 1) = (\overline{B}_m + 1)^2 - (\overline{B}_{n-1} + 1)^2.$$

Proof. Using the Binet form of $\overline{C}_n$ and the fact that $\lambda\beta = 1$, we have

$$\begin{aligned}
\overline{C}_{m+n}\overline{C}_{m-n} &= \frac{(\lambda^{m+n+1} - \beta^{m+n+1})(\lambda^{m-n+1} - \beta^{m-n+1})}{(\lambda-\beta)^2} \\
&= \frac{\lambda^{2m+2} - \lambda^{m+n+1}\beta^{m-n+1} - \beta^{m+n+1}\lambda^{m-n+1} + \beta^{2m+2}}{(\lambda-\beta)^2} \\
&= \frac{\lambda^{2m+2} - 2 + \beta^{2m+2}}{(\lambda-\beta)^2} - \frac{\lambda^{m-n+1}\beta^{m-n+1}(\lambda^{2n} + \beta^{2n}) - 2}{(\lambda-\beta)^2} \\
&= (\frac{\lambda^{m+1} - \beta^{m+1}}{\lambda-\beta})^2 - \frac{(\lambda^{2n} - 2 + \beta^{2n})}{(\lambda-\beta)^2} \\
&= (\frac{\lambda^{m+1} - \beta^{m+1}}{\lambda-\beta})^2 - (\frac{\lambda^{n} - \beta^{n}}{\lambda-\beta})^2 \\
&= \overline{C}_m^2 - \overline{C}_{n-1}^2.
\end{aligned}$$

Hence

$$(\overline{B}_{m+n} + 1)(\overline{B}_{m-n} + 1) = (\overline{B}_m + 1)^2 - (\overline{B}_{n-1} + 1)^2.$$

□

2.5 Some interesting results on co-cobalancing numbers

From section 2.4, we set $\overline{C}_n = \overline{B}_n + 1$ and in this section, let $\overline{D}_n = \sqrt{8\overline{C}_n + 1}$ where $n \in I^+$. The following theorem is similar to de-Moivre's formula see [3].

Theorem 2.8. *If n and k are natural numbers, then*

$$\left(\overline{D}_n + \sqrt{8}\overline{C}_n\right)^k = \overline{D}_{nk} + \sqrt{8}\overline{C}_{nk}.$$

Proof. From (2.8),

$$\overline{B}_{n+1} = (3\overline{B}_n + 2) + \sqrt{8\overline{B}_n^2 + 16\overline{B}_n + 9}; \quad n = 0, 1, 2, \cdots,$$

substituting,

$$\overline{C}_n = \overline{B}_n + 1,$$

thus

$$\overline{C}_{n+1} = 3\overline{C}_n + \sqrt{8\overline{C}_n^2 + 1},$$

using the Binet form, we obtain

$$\overline{D}_n^2 = 8\overline{C}_n^2 + 1 = 8\left(\frac{\lambda^{n+1} - \beta^{n+1}}{\lambda - \beta}\right)^2 + 1$$
$$= \left(\frac{\lambda^{n+1} + \beta^{n+1}}{2}\right)^2.$$

Hence

$$\overline{D}_n = \frac{\lambda^{n+1} + \beta^{n+1}}{2}.$$

Now

$$\overline{D}_n + \sqrt{8}\overline{C}_n = \frac{\lambda^{n+1} + \beta^{n+1}}{2} + \sqrt{8}\frac{\lambda^{n+1} - \beta^{n+1}}{2\sqrt{8}} = \lambda^{n+1}.$$

Thus

$$\left(\overline{D}_n + \sqrt{8}\overline{C}_n\right)^k = (\lambda^{n+1})^k = \overline{D}_{nk} + \sqrt{8}\overline{C}_{nk}.$$

□

Corollary 2.9. *If n and k are natural numbers, then*

$$\left(\overline{D}_n - \sqrt{8}\overline{C}_n\right)^k = \overline{D}_{nk} - \sqrt{8}\overline{C}_{nk}.$$

Proof. Since

$$\overline{D}_n - \sqrt{8}\overline{C}_n = \frac{\lambda^{n+1} + \beta^{n+1}}{2} - \sqrt{8}\frac{\lambda^{n+1} - \beta^{n+1}}{2\sqrt{8}} = \beta^{n+1},$$

the result follows. □

Theorem 2.10. *If m and n are natural numbers, then*

$$\overline{C}_{m+n+1} = \overline{C}_m \cdot \overline{D}_n + \overline{D}_m \cdot \overline{C}_n.$$

Proof. Since

$$(\overline{D}_m + \sqrt{8}\overline{C}_m)(\overline{D}_n + \sqrt{8}\overline{C}_n) = \lambda^{m+1} \cdot \lambda^{n+1} = \lambda^{m+n+2}$$
$$= \overline{D}_{m+n+1} + \sqrt{8}\overline{C}_{m+n+1}. \tag{2.11}$$

On the other hand,

$$(\overline{D}_m + \sqrt{8}\overline{C}_m)(\overline{D}_n + \sqrt{8}\overline{C}_n)$$
$$= \overline{D}_m\overline{D}_n + 8\overline{C}_m\overline{C}_n + \sqrt{8}(\overline{C}_m\overline{D}_n + \overline{D}_m\overline{C}_n). \tag{2.12}$$

Comparing equations (2.11) and (2.12), we get

$$\overline{D}_{m+n+1} + \sqrt{8}\overline{C}_{m+n+1} = (\overline{D}_m\overline{D}_n + 8\overline{C}_m\overline{C}_n) + \sqrt{8}(\overline{C}_m\overline{D}_n + \overline{D}_m\overline{C}_n). \quad (2.13)$$

Equating the rational and irrational parts from both sides of equation (2.13), we obtain

$$\overline{D}_{m+n+1} = \overline{D}_m \cdot \overline{D}_n + 8\overline{C}_m \cdot \overline{C}_n$$

and

$$\overline{C}_{m+n+1} = \overline{C}_m \cdot \overline{D}_n + \overline{D}_m \cdot \overline{C}_n.$$

□

Corollary 2.11. *If n is a natural number, then*

$$\overline{C}_{2n+1} = 2\overline{C}_n \cdot \overline{D}_n.$$

Proof. The proof follows by replacing m by n in Theorem 2.10. □

2.6 An application of co-cobalancing numbers to the Diophantine equation $x^2 + (x+1)^2 = y^2 + 1$

In this subsection, we derive a relation between the solutions of the Diophantine equation $x^2 + (x+1)^2 = y^2 + 1$ and the co-cobalancing numbers.

Let b be any co-cobalancing number with r its co-cobalancer and let $c = b + r$. By (2.3), we have

$$1 + 2 + \cdots + (b+1) = (b+1) + (b+2) + \cdots + c.$$

Therefore,

$$b = \frac{-2 + \sqrt{2c^2 + 2c}}{2}.$$

Thus $2c^2 + 2c$ is a perfect square. Since

$$2c^2 + 2c = c^2 + (c+1)^2 - 1,$$

we get that $x^2 + (x+1)^2 = y^2 + 1$ has the solution $x = c = b + r$ and $y = \sqrt{2c^2 + 2c}$.
For example, if we take $b = 5$, then $r = 3$ and $x = c = b + r = 8$, $y = \sqrt{2c^2 + 2c} = 12$ is the solution of the Diophantine equation $x^2 + (x+1)^2 = y^2 + 1$.
Similarly, if $b = 34$, then $r = 15$ and we have $x = 49, y = 70$ is the solution of $x^2 + (x+1)^2 = y^2 + 1$.

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