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On the Diophantine equation $a^x + (a+2)^y = z^2$, where $a \equiv_{21} 5$

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Abstract

For a fixed $a \in \mathbb{Z}^+$ with $a \equiv_{21} 5$, we show that the Diophantine equation $a^x + (a+2)^y = z^2$ has no non-negative integer solution.

1 Introduction

The Diophantine equations of the type $a^x + b^y = z^2$, where a and b are fixed, have been studied by many mathematicians. In 2020, Dokchann and Pakapongpun [1] showed that the Diophantine equation $a^x + (a+2)^y = z^2$, where a is a positive integer with $a \equiv_{42} 5$, has no non-negative integer solution. In 2022, Pakapongpun and Chattae [2] proved that $(1, 0, \sqrt{a+1})$ is the only solution (x, y, z) for the Diophantine equation $a^x + (a+2)^y = z^2$, for each fixed a such that $a \equiv_{20} 3$ and $a+1$ is a square.

In this paper, we study the Diophantine equation $a^x + (a+2)^y = z^2$ where a is a positive integer with $a \equiv_{21} 5$.

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2 Preliminaries

Throughout this paper, $a \equiv_m b$ always means a is congruent to b modulo m where a, b, m are integers such that $m \geqslant 1$. For notational convenience, we will write $a \equiv_m b, c$ to mean that $a \equiv_m b$ or $a \equiv_m c$.

Now, we shall recall the Catalan's conjecture [3] from 1844 which was proved by Mihailescu [4] in 2004.

Theorem 2.1 (Catalan's conjecture). *The Diophantine equation* $a^x - b^y = 1$ *has the unique solution* $(a, b, x, y) = (3, 2, 2, 3)$, *where* a, b, x *and* y *are integers with* $\min\{a, b, x, y\} > 1$.

Next, we give two lemmas that are consequences of the Catalan's conjecture.

Lemma 2.2. *Let* a *be a positive integer such that* $a \equiv_{21} 5$. *The Diophantine equation* $a^x + 1 = z^2$ *has no non-negative integer solution.*

Proof. Assume that there exist non-negative integers x and z such that $a^x + 1 = z^2$. If $x = 0$, then $z^2 = 2$ which is a contradiction. Now, we have $x \geqslant 1$. Since $a \geqslant 5$, by Theorem 2.1, $x = 1$. Since $a \equiv_{21} 5$, $a \equiv_7 5$. Then $z^2 \equiv_7 6$, which contradicts the fact that $z^2 \equiv_7 0, 1, 2, 4$. □

Lemma 2.3. *Let* a *be a positive integer such that* $a \equiv_{21} 5$. *The Diophantine equation* $1 + (a + 2)^y = z^2$ *has no non-negative integer solution.*

Proof. Assume that there exist non-negative integers y and z such that $1 + (a+2)^y = z^2$. If $y = 0$, then $z^2 = 2$ which is impossible. Now, we have $y \geqslant 1$. Since $a + 2 \geqslant 7$, by Theorem 2.1, $y = 1$. Since $a \equiv_{21} 5$, $a + 2 \equiv_3 1$. Then $z^2 \equiv_3 2$. This contradicts the fact that $z^2 \equiv_3 0, 1$. The proof is complete. □

3 Main Results

Now, we shall discuss a lemma which will be useful in the main theorem.

Lemma 3.1. *If* x *is a positive odd integer, then* $5^x \equiv_7 3, 5, 6$.

Proof. We prove by induction that $5^{2n-1} \equiv_7 3, 5, 6$, for all $n \in \mathbb{N}$. If $n = 1$, then $5^1 \equiv_7 5$. Thus the statement is true for $n = 1$. Assume that it is true for $n = k$. Then $5^{2k-1} \equiv_7 3, 5, 6$ and so $5^{2k+1} \equiv_7 5, 6, 3$. Hence, the statement is true for $n = k + 1$ which proves the result. □

Next, we give our main result.

Theorem 3.2. *Let a be a positive integer such that $a \equiv_{21} 5$. The Diophantine equation $a^x + (a+2)^y = z^2$ has no non-negative integer solution where x, y, z are non-negative integers.*

Proof. Assume that there exist non-negative integers x, y, z such that $a^x + (a+2)^y = z^2$. By Lemma 2.2 and Lemma 2.3, $x \geqslant 1$ and $y \geqslant 1$. If x is even, then $a^x \equiv_3 1$ because $a \equiv_{21} 5$. Since $(a+2)^y \equiv_3 1$, $z^2 \equiv_3 2$ which contradicts the fact that $z^2 \equiv_3 0, 1$. As a result, x is odd. Since $a \equiv_7 5$, by Lemma 3.1, we have $a^x \equiv_7 3, 5, 6$. Since $(a+2)^y \equiv_7 0$, we obtain $z^2 \equiv_7 3, 5, 6$, which contradicts the fact that $z^2 \equiv_7 0, 1, 2, 4$. This completes the proof. □

4 Conclusion

In this paper, we proved that the Diophantine equation $a^x + (a+2)^y = z^2$, where a is a positive integer such that $a \equiv_{21} 5$ has no non-negative integer solution.
Note that the main theorem in [1] is a special case of Theorem 3.2.

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