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The a-fuzzy normed algebra and its basic properties

Rasha Khudhur Abbas, Jehad Ramadan Kider

Branch of Mathematics and Computer Applications
Department of Applied Sciences
University of Technology
Baghdad, Iraq

email: as.21@grad.uotechnology.edu.iq, jehad.r.kider@uotechnology.edu.iq

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Abstract

In this article, we introduce the notion of a-fuzzy normed algebra by using two binary operations: the $t-$conorm $\circledast$ defined as $\mu \circledast \omega = \mu + \omega - \mu\omega$ for all $\mu, \omega \in [0,1]$ and the $t-$norm $\odot$ defined as $\eta \odot \theta = \eta.\theta$ for all $\eta, \theta \in [0,1]$. Moreover, we give some examples to show the existence of such a notion. Furthermore, we introduce basic properties of a fuzzy complete a-fuzzy normed algebra and prove that $\odot$ is a fuzzy continuous function and that every a-fuzzy normed algebra Z can be embedded in $afb(Z, Z)$ as a closed subalgebra.

1 Introduction

This research consists of two sections:
In section 2, we define the a-fuzzy normed space and study its basic properties. Then we introduce theorems that are needed for section 3.
In section 3, we introduce the definition of a-fuzzy normed algebra and prove some important theorems of fuzzy complete a-fuzzy normed algebra.

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2 Concepts and theorems for a-fuzzy normed spaces

For the definition of t-norm and t-conorm and their important properties, we refer the interested reader to [1] and [2], respectively.

Definition 2.1. *[3] If* $a_{\mathbb{R}} : \mathbb{R} \to I$ *is a fuzzy set and* $\circledast$ *is a* t*-conorm, then* $a_{\mathbb{R}}$ *is an a-fuzzy absolute value on* $\mathbb{R}$ *if:*
(i) $0 < a_{\mathbb{R}}(\mu) \leq 1$.
(ii) $a_{\mathbb{R}}(\mu) = 0$ *if and only if* $\mu = 0$.
(iii) $a_{\mathbb{R}}(\eta\mu) \leq a_{\mathbb{R}}(\eta).a_{\mathbb{R}}(\mu)$.
(iv) $a_{\mathbb{R}}(\eta + \mu) \leq a_{\mathbb{R}}(\eta) \circledast a_{\mathbb{R}}(\mu)$
for all $\eta, \mu \in \mathbb{R}$.
In this case, $(\mathbb{R}, a_{\mathbb{R}}, \circledast)$ *is an* **a-fuzzy absolute value space.**

Definition 2.2. *Let* $L_{\mathbb{C}} : \mathbb{C} \to I$ *be a fuzzy set and let* $\circledast$ *be a* t*-conorm. Then* $L_{\mathbb{C}}$ *is an a-fuzzy length on* $\mathbb{C}$ *if:*
(i) $0 < L_{\mathbb{C}}(\sigma) \leq 1$.
(ii) $L_{\mathbb{C}}(\sigma) = 0$ *if and only if* $\sigma = 0$.
(iii) $L_{\mathbb{C}}(\sigma\tau) \leq L_{\mathbb{C}}(\sigma).L_{\mathbb{C}}(\tau)$.
(iv) $L_{\mathbb{C}}(\sigma + \tau) \leq L_{\mathbb{C}}(\sigma) \circledast L_{\mathbb{C}}(\tau)$ *for all* $\sigma, \tau \in \mathbb{C}$.
In this case, $(\mathbb{C}, L_{\mathbb{C}}, \circledast)$ *is an* **a-fuzzy length space.**

Remark 2.3. *We will take* $\circledast$ *to be* $\alpha \circledast \beta = \alpha + \beta - \alpha\beta \ \forall \alpha, \beta \in I$.

Example 2.4. *[3] Let* $a_{|.|}(\alpha) = \frac{|\alpha|}{1+|\alpha|}$ *for all* $\alpha \in \mathbb{R}$ *where* $|.|$ *is the absolute value on* $\mathbb{R}$*. Then* $(\mathbb{R}, a_{|.|}, \circledast)$ *is an a-fuzzy absolute value space.*

Example 2.5. *Let* $L_{|.|}(\alpha) = \frac{|\alpha|}{1+|\alpha|}$ *for all* $\alpha \in \mathbb{C}$ *where* $|.|$ *is the length value on* $\mathbb{C}$*. Then* $(\mathbb{C}, a_{|.|}, \circledast)$ *is an a-fuzzy length space.*

Definition 2.6. *[3] Let* $(\mathbb{C}, L_{\mathbb{C}}, \circledast)$ *be an a-fuzzy length space and let* Z *be a vector space over* $\mathbb{C}$*. Suppose that* $\circledast$ *is a* t*-conorm and* $n_Z : Z \to I$ *is a fuzzy set. Then* n_Z *is an a-fuzzy norm on* Z *if:*
(i) $0 < n_Z(z) \leq 1$.
(ii) $n_Z(z) = 0 \Leftrightarrow z = 0$.
(iii) $n_Z(\mu z) \leq L_{\mathbb{C}}(\mu)n(z)$ *for all* $0 \neq \mu \in \mathbb{C}$.
(iv) $n_Z(z + y) \leq n_Z(z) \circledast n_Z(y) \ \forall z, y \in Z$.
Here, we say that $(Z, n_Z, \circledast)$ *is an* **a-fuzzy normed space** *(or simply a-FNS).*

Example 2.7. *[3] Define* $n_{\|.\|}(z) = \frac{\|z\|}{(1+\|z\|)}$, $\forall z \in Z$. *Then* $(Z, n_{\|.\|}, \circledast)$ *is a-FNS if* $(Z, \|.\|)$ *is a normed space. Also* $n_{\|.\|}$ *is called the* **standard a-fuzzy norm on** Z.

Definition 2.8. *[3] Suppose that* $(Z, n_Z, \circledast)$ *is an a-FNS. If* (z_k) *is a sequence in Z, then* (z_k) *is said to be* **fuzzy convergent** *to the limit* z *as* $k \to \infty$ *if* $\forall \mu \in (0,1), \exists N \in \mathbb{N}$ *such that* $n_Z(z_k - z) < \mu$, *for all* $k \geq N$. *If* (z_k) *is fuzzy convergent to* z, *then we write* $\lim_{k\to\infty} z_k = z$ *or* $z_k \to z$ *as* $k \to \infty$ *or* $\lim_{k\to\infty} (z_k - z) = 0$.

Definition 2.9. *[3] Suppose* $(Z, n_Z, \circledast)$ *is an a-FNS. A sequence* (z_k) *is a* **fuzzy Cauchy** *sequence in Z if* $\forall\ \mu \in (0,1), \exists\ N \in \mathbb{N}$ *such that* $n_Z(z_k - z_m) < \mu$, $\forall k, m \geq N$.

Definition 2.10. *[3] If for all fuzzy Cauchy sequences* (z_k) *in* Z, $\exists\ z \in Z$ *such that* $z_k \to z$, *then the a-fuzzy normed space* $(Z, n_Z, \circledast)$ *is said to be* **fuzzy complete.**

Theorem 2.11. *[4] The a-fuzzy absolute space* $(\mathbb{R}, a_{\mathbb{R}}, \circledast)$ *is fuzzy complete.*

Theorem 2.12. *[4] If* $(Z, n, \circledast)$ *is an a-FNS, then* $(Z^k, n_k, \circledast)$ *is a fuzzy complete a-FNS if and only if* $(Z, n, \circledcirc)$ *is fuzzy complete, where* $Z^k = Z \times Z \times \times Z$ *[k-times],* $k \in \mathbb{N}$, *and* $n_k[(z_1, z_2, ..., z_k)] = n(z_1) \circledast n(z_2) \circledast, ..., \circledast n(z_k)$ *for all* $(z_1, z_2, ..., z_k) \in Z^k$.

Corollary 2.13. $(\mathbb{R}^k, n_k, \circledast)$ *is fuzzy complete.*

Corollary 2.14. *The a-fuzzy length space* $(\mathbb{C}, L_{\mathbb{C}}, \circledast)$ *is fuzzy complete.*

Proof. Since $\mathbb{C} = \mathbb{R}^2$, it follows that $(\mathbb{C}, L_{\mathbb{C}}, \circledast)$ is fuzzy complete. □

Theorem 2.15. *[4] The operator* $H : Z \to W$ *is fuzzy continuous at* $z \in Z$ *if and only if whenever* (z_k) *is fuzzy convergent to* $z \in Z$, *then* $(H(z_k))$ *is fuzzy convergent to* $H(z) \in W$.

Theorem 2.16. *[4] If* $(Z, n_1, \circledcirc)$ *is an a-FNS, then the a-fuzzy norm* n_2 *is equivalent to* n_1 *if* $\exists\ p, q$ *in* $(0,1)$ *with* $pn_2(z) \leq n_1(z) \leq qn_2(z)$.

Definition 2.17. *[4] Suppose that* $(Z, n_Z, \circledast)$ *and* $(Y, n_Y, \circledast)$ *are two a-FNS. The operator* $S : D(S) \to Y$ *is said to be* **fuzzy bounded** *if* $\exists\ \mu \in (0,1)$ *such that* $n_Y[S(z)] < \mu n_Z(z)$ *for all* $z \in D(S)$.

Notation 2.18. *[4] Suppose that* $(Z, n_Z, \circledast)$ *and* $(Y, n_Y, \circledast)$ *are two a-FNS. We use the notation* $afb(Z, Y) = \{S : Z \to Y\}$ *for a fuzzy bounded operator.*

Theorem 2.19. *[4] Define:* $n_{afb(Z,Y)}(S) = sup_{z\in D(S)} n_W(Sz), \forall\ S \in afb(Z,Y)$. *Then* $[afb(Z,Y), n_{afb(Z,W)}, \circledast]$ *is a-FNS if* $(Z, n_Z, \circledast)$ *and* $(Y, n_Y, \circledast)$ *are two a-FNS.*

Theorem 2.20. *[4] Suppose that* $(Z, n_Z, \circledast)$ *and* $(Y, n_Y, \circledast)$ *are two a-FNS. If* Y *is fuzzy complete, then* $afb(Z,Y)$ *is fuzzy complete .*

Definition 2.21. *[4] A linear functional* h *from a-FNS* $(Z, n_Z, \circledast)$ *into the a-fuzzy absolute space* $(\mathbb{R}, a_{\mathbb{R}}, \circledast)$ *is said to be a* **fuzzy bounded functional** *if there exists* $s \in (0,1)$ *such that* $a_{\mathbb{R}}[h(u)] < s.n_U(u)$ *for any* $u \in D(h)$. *Furthermore, the a-fuzzy norm of* h *is* $n_{afb(Z,\mathbb{R})}(h) = sup_{u\in D(L)} a_{\mathbb{R}}(hu)$ *for all* $L \in afb(Z,\mathbb{R})$ *and* $a_{\mathbb{R}}[h(u)] < n_{afb(Z,\mathbb{R})}(h).n_Z(u)$ *for any* $u \in D(h)$.

Definition 2.22. *[4] Let* $(Z, n_z, \circledast)$ *be an a-FNS. Then* $afb(Z,\mathbb{R}) = \{h : Z \to \mathbb{R}\}$, *where* h *is fuzzy bounded and linear and forms a-fuzzy normed space with the a-fuzzy norm defined by* $n_{afb(Z,R)}(h) = sup_{u\in D(L)} a_R(hu)$. *Here,* $afb(Z,\mathbb{R}) = \{h : Z \to \mathbb{R}\}$ *is called the* **fuzzy dual space** *of* Z.

Theorem 2.23. *[4] If* $(Z, n_Z, \circledast)$ *is an a-FNS, then the fuzzy dual space* $afb(Z,\mathbb{R})$ *is fuzzy complete.*

Definition 2.24. *[4] Suppose that* Z *is a vector space over the field* $\mathbb{K}$ *and* D *is a closed subspace of* Z. *Then* $\frac{Z}{D} = \{z + D : z \in Z\}$ *is a vector space over the field* $\mathbb{K}$ *with the operations:* $(v + D) + (z + D) = (v + z) + D$ *and* $\alpha(z + D) = (\alpha z) + D$.

Definition 2.25. *[5] Suppose that* $(Z, n_Z, \circledast)$ *is an a-FNS and* $D \subset Z$ *is fuzzy closed in* Z. *Define a-fuzzy norm for the quotient space* $\frac{Z}{D}$ *by* $q[u+D] = inf_{d\in D} n_U[z + d]$ *for all* $z + D \in \frac{Z}{D}$.

Theorem 2.26. *[5] The quotient space* $(\frac{Z}{D}, q, \circledast)$ *is an a-FNS if* $(Z, n_Z, \circledast)$ *is an a-FNS and* $D \subset Z$ *is fuzzy closed in* Z.

Remark 2.27. *[5] If* $(Z, n_Z, \circledast)$ *is a-FNS and* $D \subset Z$ *is fuzzy closed in* Z, *then*
(1) $\pi : Z \to \frac{Z}{D}$ *is a natural operator defined by* $\pi[z] = z + D$.
(2) $q(z + D) \leq n_Z(z)$.

Theorem 2.28. *[5] Suppose that* $(Z, n_Z, \circledast)$ *is an a-FNS and* $D \subset Z$ *is fuzzy closed in* Z. *If* $(\frac{Z}{D}, q, \circledast)$ *is fuzzy complete, then* $(Z, n_Z, \circledcirc)$ *is fuzzy complete.*

Theorem 2.29. *[5] Suppose that* $(Z, n_Z, \circledast)$ *is a-FNS and* $D \subset Z$ *is fuzzy closed in* Z. *If* $(Z, n_Z, \circledast)$ *is fuzzy complete, then* $(\frac{Z}{D}, q, \circledcirc)$ *is fuzzy complete.*

Theorem 2.30. *[4] Let $(Z, n_Z, \circledcirc)$ be a-fuzzy normed space. The geometric series $\sum_{j=0}^{\infty} z^j = 1 + z +^2 +... + z^k + ...,$ is fuzzy convergent with sum $\frac{1}{1-z}$ whenever $n_Z(z) < 1$, and diverge whenever $n_Z(z) \geq 1$.*

3 When the a-fuzzy normed algebra is fuzzy complete

Definition 3.1. *The space $(Z, n_Z, \circledast, \odot)$ is called an* **a-fuzzy normed algebra space** *(or simply a- FNAS) if*
(1) $(Z, +, .)$ is an algebra space over the field K, where $K = R$ or $K = C$.
(2) $(Z, n_Z, \circledast)$ is an a-FNS, with $\circledast$ a continuous t-conorm.
(3) $\odot$ is a continuous t-norm.
(4) $n_Z(a.b) \leq n_Z(a) \odot n_Z(b)$ for all $a, b \in Z$.

Remark 3.2. *Here, we take*
(1) $\sigma \odot \tau = \sigma.\tau,\ \forall\, \sigma, \tau \in [0, 1]$.
(2) $\gamma \circledast \delta = \gamma + \delta - \gamma\delta,\ \forall\, \gamma, \delta \in [0, 1]$.

Example 3.3. *If $(Z, +, .)$ is an algebra, then $(Z, n_Z, \circledast, \odot)$ is an a-FNAS, with* $n_Z(u) = \begin{cases} 0 \text{ if } u = 0 \\ 1 \text{ if } u \neq 0 \end{cases}$
which is called the **discrete a-FNAS.**

Proof.
(1) It is clear that $(Z, n_Z, \circledast)$ is an a-FNS.
(2) We have to prove that $n_Z(uv) \leq n_Z(u) \odot n_Z(v)$ for all $u, v \in Z$.
Case 1: if $u = 0$ and $v = 0$, then $u.v = 0$ with $0 \odot 0 = 0$ and so the inequality holds.
Case 2: if $u \neq 0$ and $v \neq 0$, then $u.v \neq 0$ with $1 \odot 1 = 1$ and so the inequality follows.
Case 3: if $u = 0$ or $v = 0$, then $u.v = 0$ with $1 \odot 0 = 0 \odot 1 = 0$ and so the inequality obtains. □

Example 3.4. *Define $n_Z(u) = \frac{\|u\|}{1+\|u\|}$ for all $u \in Z$. If $(Z, \|\cdot\|, +, .)$ is a normed algebra, then $(Z, n_Z, \circledast, \odot)$ is an a-FNAS.*

Proof.
(1) By example 2.7, we have $(Z, n_Z, \circledast)$ is an a-FNS.

(2) $n_Z(u) \odot n_Z(v) = n_Z(u).n_Z(v) = [\frac{\|u\|}{1+\|u\|}].[\frac{\|v\|}{1+\|v\|}] = \frac{\|u\|\|v\|}{[1+\|u\|[1+\|v\|]} \geq \frac{\|uv\|}{1+\|uv\|} = n_Z(uv)$, since $1 + \|uv\| < [1 + \|u\|].[1 + \|v\|]$ and $\|uv\| < \|u\| . \|v\|$. □

Definition 3.5. *If $(Z, n_Z, \circledast)$ is a fuzzy complete a-FNS, then $(Z, n_Z, \circledast, \odot)$ is called a* **fuzzy complete a-FNA.**

Example 3.6. *Let $Z = C[0, 1]$ with $n_Z(f) = sup_{x\in[0,1]}a_{\mathbb{R}}(f(x))$. Let $\odot$ be defined on Z pointwise as follows:*
$(f \odot g)(z) = (f.g)(z) = f(z).g(z) = f(z) \odot g(z)$.
Then $(Z, n_Z, \circledast, \odot)$ is a commutative fuzzy complete a-FNA.

Proof.
(1) By example 2.15 in [5], we have $(Z, n_Z, \circledast)$ is an a-FNS.
(2) To show that $n_Z(f.g) \leq n_Z(f) \odot n_Z(g)$, $n_Z(f.g) = sup_{x\in[0,1]}a_{\mathbb{R}}(f.g(x)) = sup_{x\in[0,1]}a_{\mathbb{R}}(f(x).g(x)) \leq [sup_{x\in[0,1]}a_{\mathbb{R}}(f(x))].[sup_{x\in[0,1]}a_{\mathbb{R}}(g(x))] = n_Z(f) \odot n_Z(g)$. Hence $(Z, n_Z, \circledast, \odot)$ is a commutative fuzzy complete a-FNA. □

Example 3.7. *Let D denote the closed unit disc in $\mathbb{C}$ and let Z denote the set of fuzzy continuous complex valued functions on D which are analytic in the interior of D. Equip Z with pointwise addition and multiplication and the a-fuzzy norm $n_Z(f) = sup\{L_{\mathbb{C}}(fz) : z \in \partial D\}$, where ∂D is the boundary of D. Then $(Z, n_Z, \circledast, \odot)$ is a fuzzy complete a-FNA and it is commutative with identity. Here, $(Z, n_Z, \circledast, \odot)$ is called* **the disc a-FNA.**

Lemma 3.8. *If $(Z, n_Z, \circledast, \odot)$ is an a-FNA, then multiplication is a fuzzy continuous function.*

Proof.
If (z_k) and (u_k) are sequences in Z, with $z_k \to z$ and $u_k \to u$ as $k \to \infty$, then for any given $0 < \gamma < 1$ and $0 < \alpha < 1$ there is M such that $n_Z(z_k - z) \leq \alpha$, for all $k \geq M$ and $n_Z(u_k - u) \leq \gamma$, $\forall\ k \geq M$. Put $n_Z(z_k) = \beta_k$ and $n_Z(u) = \sigma$, for some $0 < \beta_k, \sigma < 1$. In addition, let $\beta_k \odot \sigma < \varepsilon$ and $\alpha \odot \beta < \delta$, for some $0 < \delta, \varepsilon < 1$. Now $n_Z(z_k u_k - zu) = n_Z(z_k u_k - z_k u + z_k u - zu) \leq n_Z(z_k(u_k - u)) \circledast n_Z((z_k - z)u)) \leq n_Z(z_k) \odot n_Z(u_Z - u) \circledast n_Z(z_k - z) \odot n_Z(u) < (\beta_k \odot \gamma) \circledast (\alpha \odot \sigma) < \varepsilon \circledast \delta$. Choose $0 < \mu < 1$ satisfying $\varepsilon \circledast \delta < \mu$. Thus $n_Z(z_k u_k - zu) < \mu$, for all $k \geq M$. Therefore, $z_k u_k \to zu$ as $k \to \infty$ and hence multiplication is fuzzy continuous. □

Remark 3.9. *It is clear that $\sigma \odot \mu \leq \rho \circledast \mu$, for all σ, $\mu \in [0, 1]$.*

Theorem 3.10. *An a-FNA $(Z, n_Z, \circledast, \odot)$ without identity can be embedded into an a-FNA Z_e with identity e and Z is an ideal in Z_e.*

Proof.
Put $Z_e = Z \times \mathbb{C}$ and define multiplication in Z_e by $(z, \alpha).(u, \beta) = (zu + \beta z + \alpha u, \alpha\beta)$. Then Z_e is an algebra with identity $\acute{e} = (0, 1)$ since $(a, \alpha).(0, 1) = (a, \alpha)$, for all $(a, \alpha) \in Z_e$. Then Z_e is an a-FNS with an a-fuzzy norm $n_{Z_e} : Z_e \to [0, 1]$ defined by: $n_{V_e}(z, \alpha) = n_Z(z) \circledast L_{\mathbb{C}}(\alpha)$. Now $n_{Z_e}(z, \alpha).(u, \beta) = n_{Z_e}(zu + \beta z + \alpha u, \alpha\beta) = n_Z(zu + \beta z + \alpha u) \circledast L_{\mathbb{C}}(\alpha\beta) \leq n_Z(zu) \circledast L_{\mathbb{C}}(\alpha\beta)$ $\leq [n_Z(z) \odot n_Z(u)] \circledast [L_{\mathbb{C}}(\alpha) \odot L_{\mathbb{C}}(\beta)] \leq [n_Z(z) \circledast n_Z(u)] \circledast [L_{\mathbb{C}}(\alpha) \circledast L_{\mathbb{C}}(\beta)]$ $\leq [n_Z(z) \circledast L_{\mathbb{C}}(\alpha)] \circledast [n_Z(u) \circledast L_{\mathbb{C}}(\beta)] \leq n_{Z_e}(z, \alpha) \circledast n_{Z_e}(u, \beta)$. □

Proposition 3.11. *The space $(Z_e, n_{Z_e}, \circledast, \odot)$ is fuzzy complete if and only if $(Z, n_Z, \circledast, \odot)$ is fuzzy complete.*

Proof.
Suppose that $Z_e = Z \times \mathbb{C}$ is fuzzy complete and let (z_k) and (α_k) be fuzzy Cauchy sequences in Z and $\mathbb{C}$, respectively; that is, for any given $0 < \varepsilon < 1, 0 < \sigma < 1$, there exist M_1 and M_2 such that $n_Z(z_k - z_m) < \varepsilon$, for all $k, m \geq M_1$ and $L_{\mathbb{C}}(\alpha_k - \alpha_m) < \sigma$, for all $k, m > M_2$. Let $M = min\{M_1, M_2\}$. Now, $n_{Z_e}[(z_k, \alpha_k) - (z_m, \alpha_m)] = n_{Z_e}(z_k - z_m, \alpha_k - \alpha_m) = n_Z[z_k - z_m] \circledast L_{\mathbb{C}}[\lambda_n - \lambda_m, t] < \varepsilon \circledast \sigma$. Choose $0 < \mu < 1$ with $\varepsilon \circledast \sigma < \mu$. Then $n_{Z_e}[(z_k, \alpha_k) - (z_m, \alpha_m)] < \mu$ for all $k, m > M$. Thus $\{(z_k, \alpha_k)\}$ is a fuzzy Cauchy sequence in Z_e. But Z_e is fuzzy complete. So there is $(z, \alpha) \in Z_e$ such that $(z_k, \alpha_k) \to (z, \alpha)$ as $k \to \infty$; that is,
$0 = \lim_{k\to\infty} n_{Z_e}[(z_k, \alpha_n) - (z, \alpha)] = \lim_{k\to\infty} n_Z(z_k - z) \circledast \lim_{k\to\infty} a_{\mathbb{C}}(\alpha_k - \alpha)$.
Therefore, $\lim_{k\to\infty} n_Z(z_k - z) = 0$ and $\lim_{k\to\infty} L_{\mathbb{C}}(\alpha_k - \alpha) = 0$. Hence Z is fuzzy complete.
Conversely, assume that Z is fuzzy complete and let $\{(z_k, \alpha_k)\}$ be a fuzzy Cauchy sequence in $Z_{e.}$ Then for any given $0 < \varepsilon < 1$, there is M such that $n_{Z_e}[(z_k, \alpha_k) - (z_m, \alpha_m)] < \varepsilon$, for all $k, m > M$ or $n_Z(z_k - z_m) \circledast L_{\mathbb{C}}(\alpha_k - \alpha_{\acute{m}} t) < \varepsilon$. Hence $n_Z(z_k - z_m) < \varepsilon$ and $a_{\mathbb{C}}(\alpha_k - \alpha_{\acute{m}} t) < \varepsilon$, for all $k, m > M$. This implies that (z_k) and (α_k) are fuzzy Cauchy sequences in Z and $\mathbb{C}$, respectively. But Z and $\mathbb{C}$ are fuzzy complete. So there is $z \in Z$ and $\alpha \in \mathbb{C}$ such that $\lim_{k\to\infty} n_Z(z_k - z) = 0$ and $\lim_{k\to\infty} L_{\mathbb{C}}(\alpha_k - \alpha) = 0$. Now, $\lim_{k\to\infty} n_{V_e}[(z_k, \alpha_k) - (z, \alpha)] = \lim_{k\to\infty} n_Z(z_k - z) \circledast \lim_{k\to\infty} L_{\mathbb{C}}(\alpha_n - \alpha) = 0 \circledast 0 = 0$. Hence $(z_k, \alpha_k) \to (z, \alpha)$ as $k \to \infty$. Consequently, Z_e is fuzzy complete. □

Theorem 3.12. *Let $(Z, n_Z, \circledast, \odot)$ be an a-FNA with identity e. Then there is a a-fuzzy norm $\acute{n}_Z$ on Z such that n_Z is equivalent to $\acute{n}_Z$ implies $(Z, \acute{n}_Z, \circledast, \odot)$ is an a-FNA with $\acute{n}_Z(e) = 1$.*

Proof.
For each $x \in Z$, let N_x be a linear operator defined by $N_x(z) = xz$, for all

$z \in Z$. If $N_x = N_y$, it follows that $N_x(e) = N_y(e)$. So $x = y$ and, as a result, the operator $x \longmapsto N_x$ is injective from Z into the set of all linear operators on Z. Since $n_Z(N_x(z)) = n_Z(xz) \leq n_Z(x) \circledast n_Z(v)$ for $z \in Z$, we have N_x is fuzzy bounded and $n_Z(N_x) \leq tn_Z(x)$. Put $\acute{n}_Z(z) = n_Z(N_{(z,)})$. Then

$$\acute{n}_Z(z) \leq t.n_Z(z)..........................(1)$$

for some $t, 0 \leq t \leq 1$.
On the other hand,

$$\acute{n}_Z(z) = n_Z(N_z) = sup_{y\in D(N_z)} n_Z(N_z(y)) = sup_{y\in D(N_z)} n_Z(zy)$$
$$\geq n_Z(z\acute{y}) = n_Z(z) \odot n_Z(\acute{y}) = n_Z(z).n_Z(\acute{y}),$$

or

$$\acute{n}_Z(z) \geq sn_Z(z)....................(2)$$

for some s such that $0 \leq s \leq 1$.
From (1) and (2) we have $sn_Z(z) \leq \acute{n}_Z(z) \leq tn_Z(z)$ for all $z \in Z$. Hence $n_Z(.)$ is equivalent to $\acute{n}_Z(.)$. Now $\acute{n}_Z(uw) = n_Z(N_{uw}) = n_Z(N_u.N_w) \leq n_Z(N_u) \odot n_Z(N_w) \leq \acute{n}_Z(u) \odot \acute{n}_Z(w)$. Therefore, $(Z, \acute{n}_Z, \circledast, \odot)$ is an a-FNA. Consequently, $\acute{n}_Z(e) = n_Z(N_e) = 1$. □

Theorem 3.13. *Every a-FNA $(Z, n_Z, \circledast, \odot)$ can be embedded as a closed subalgebra of $afb(Z, Z)$.*

Proof.
Define $N_z : Z \to Z$ by $N_z(u) = zu$, for all $u \in Z$. Then $N_z \in afb(Z, Z)$ since $N_z(u_1 + u_2) = zu_1 + zu_2 = N_z(u_1) + N_z(u_2)$ and $N_z(\alpha u_1) = z(\alpha u_1) = \alpha(zu_1) = \alpha N_z(u_1)$. Also, $n_Z(N_z(u)) = n_Z(zu) \leq n_Z(z) \odot n_Z(u) \leq n_Z(z)$; that is, $n_Z(N_z) \leq n_Z(z)$ Now we show that $N_{a+b} = N_a + N_b$ and $N_{ab} = N_a.N_b$, $N_{\alpha a} = \alpha N_a$, as well as $N_e = I_Z$. We have $N_{a+b}(z) = (a + b)z = az + bz = N_a(z) + N_b(z)$ and $N_{\alpha a}(z) = (\alpha a)z = \alpha(az) = \alpha N_a(z)$. In addition, $N_{ab}(z) = (ab)z = a(bz) = N_a.N_a(z)$ as well as $N_e(z) = ez = z = I_Z(z)$. We have $n_Z(N_x(y)) = n_Z(xy) \leq n_Z(x) \odot n_Z(y) = n_Z(x).n_Z(y)$. Put $n_Z(x) = \delta$, for some $0 < \delta < 1$. That is, $n_Z(N_x(y)) \leq \delta.n_Z(y)$. Therefore, N_x is fuzzy bounded. Let $T : Z \to afb(Z, Z)$ be a mapping defined by $T(z) = N_z$. T isometric and so it is injective. Moreover, the image of the operator T $T(Z) = \{N_z : z \in Z\}$ is a subalgebra of $afb(Z, Z)$ and $T(Z)$ is fuzzy closed in $afb(Z, Z)$. Now, suppose that N_{z_k} is a sequence in $afb(Z, Z)$ such that $N_{z_k} \to S$ in $afb(Z, Z)$. Then $N_{z_k}(x) = z_k x = N_e(z_k)x$ and so, as $k \to \infty$, $S(x) = S(ex)$; that is, $S = N_e$. Thus $T(Z)$ is fuzzy closed in $afb(Z, Z)$. □

Proposition 3.14. *If $(Z, n_Z, \circledast, \odot)$ is a fuzzy complete a-FNA and $z \in Z$, then $e - z$ is invertible and the series $\sum_{k=0}^{\infty} z^k$ is fuzzy convergent where $\sum_{k=0}^{\infty} z^k = (e - z)^{-1}$.*

Proof.
Let $z \in Z$. Put $s_k = 1 + z + z^2 + ... + z^k$ or $s_k = \sum_{j=0}^{k} z^j$. Then s_k is a fuzzy Cauchy sequence in Z and so it is fuzzy convergent since Z is complete. Let u denote its limit; that is, $u = \sum_{j=0}^{\infty} z^j$. We will prove that u is the inverse of $e - z$ as follows:
$(e - z)u = \lim_{k\to\infty}(e - z)s_k = \lim_{k\to\infty}(e - z^{k+1}) = e$, and $u(e - z) = \lim_{k\to\infty} s_k(e-z) = \lim_{k\to\infty}(e-z^{k+1}) = e$. Hence $(e-z)u = u(e-z) = e$. □

Theorem 3.15. *Let $(Z, n_Z, \circledast, \odot)$ be a fuzzy complete a-FNA and suppose that D is a fuzzy closed ideal in Z. Then $(\frac{Z}{D}, q, \circledast, \odot)$ is a fuzzy complete a-FNA. If Z has identity, then so does $\frac{Z}{D}$. Moreover, the identity of $\frac{Z}{D}$ has fuzzy norm equal to 1.*

Proof.
We know that $\frac{Z}{D}$ is a fuzzy complete a-FNS by Theorem 2.29. Since D is an ideal, it is easy to see that $\frac{Z}{D}$ is an algebra with multiplication given by $(x + D)(y + D) = (xy) + D$. Now, $q[(x + D).(y + D)] = q[(xy) + D] = inf_{d\in D} n_Z[(xy + d)] \leq inf_{d\in D} n_Z[(x + d).(y + d)] \leq inf_{d\in D} n_Z(x + d) \odot inf_{d\in D} n_Z(y+d) = q(x+D) \odot q(y+D)$. Thus $(\frac{Z}{D}, q, \circledast, \odot)$ is a fuzzy complete a-FNA. Moreover, if e is the identity of Z with $n_Z(e) = 1$, then $e + D$ is the identity of $\frac{Z}{D}$. Furthermore, $q(e + D) = inf_{d\in D} n_Z(e + d) = n_Z(e) = 1$. □

Remark 3.16. *If $(Z, n_Z, \circledast, \odot)$ is fuzzy complete, then for any $a \neq 0, a^{-1}$ exists and $a^{-1} \in Z$.*

Proof.
If $0 \neq a \in Z$, then we put $a = e - (e - a)$. Using Proposition 3.14 with $z = e - a$, we see that a is invertible and its inverse a^{-1} is given by the convergent series $\sum_{k=0}^{\infty}(e - a)^k$. □

Proposition 3.17. *If $(Z, n_Z, \circledast, \odot)$ is a fuzzy complete a-FNA, then the inverse operator $z \to z^{-1}$ is a fuzzy continuous mapping.*

Proof.
First, we show that the inverse map is fuzzy continuous at e. Let $0 < \varepsilon < 1$ be given. We want to find $0 < \delta < 1$ such that $n_Z(u - e) < \delta$ implies $n_Z(u^{-1}e) < \varepsilon$. Since $n_Z(u - e) < 1$ implies $u^{-1} = \sum_{k=0}^{\infty}(e - u)^k$, we have $n_Z(u^{-1}e) = n_Z(\sum_{k=1}^{\infty}(e - u)^k) \leq \delta \circledast \delta^2 \circledast \delta^3...$ Setting $\delta \circledast \delta^2 \circledast \delta^3... < \varepsilon$, we

get $n_Z(u^{-1} - e) < \varepsilon$. Now, as $n \to \infty$, $z_n \to z$ and so $z_n z^{-1} \to zz^{-1} = e$. This implies that $(z_n z^{-1})^{-1} \to e$ or $zz_n^{-1} \to e$. Consequently, $z_n^{-1} \to z^{-1}$ as $n \to \infty$. □

Lemma 3.18. *Let $(Z, n_Z, \circledast, \odot)$ be fuzzy complete with identity e. If z and u are invertible elements of Z, then zu and uz are invertible.*

Proof.
We have $n_z(uz) \leq n_z(u) \odot n_z(z)$, $n_z(uz) \leq 1$. Similarly, $n_z(zu) \leq 1$. This implies that $e - uz$ and $e - zu$ are both invertible with inverses given as $a = (e - uz)^{-1} = \sum_{k=0}^{\infty}(uz)^k$ and $b = (e - zu)^{-1} = \sum_{k=0}^{\infty}(zu)^k$, respectively. □

Proposition 3.19. *Let $(Z, n_Z, \circledast, \odot)$ be a fuzzy complete a-FNA with identity e. Suppose that z and u are elements of Z such that $e - zu$ is invertible. Let $a = (e - zu)^{-1.}$ Then $b = e + uaz$ is the inverse of $e - uz$.*

Proof.
$b(e-uz) = (e+uaz)(e-uz) = e-uz+uaz-uazuz = e-uz+ua(e-zu)z = e - uz + u[a(e - zu)]z = e - uz + uz = e$. □

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