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Some New Results on Hourglass Matrices

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Abstract

An hourglass matrix is a nonsingular matrix obtained from quadrant interlocking factorization called WH factorization. In this paper, we establish the determinant of hourglass matrix and show its application in the triangular blocks of hourglass matrix called H_{system}. Therefore, WH factorization exists for every nonsingular matrix and hence its Schur complement exists for every H_{system}.

1 Introduction

The components of a matrix factorization are of prime importance but not just as a mechanism for solving another problem [19]. Matrix factorization, such as quadrant interlocking factorization (QIF), serves to decompose original task that may be relatively difficult to solve into subtasks which are

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solved and regrouped. QIF, an alternate to LU factorization but commonly known as WZ factorization, is a factorization technique used to break nonsingular matrices into block forms, assembled and then solved as sub-blocks, see [15, 18, 11]. WZ factorization of a matrix B produces interlocking quadrant factors called W-matrix and Z-matrix such that [8]

$$B = WZ. \tag{1.1}$$

WZ factorization requires $\sum_{k=1}^{\lfloor \frac{n}{2}-1 \rfloor} (n-2k)$ of linear systems which can be computed via Cramer's rule adopting the least condition number [4]. The factorization is more effective for real symmetric, diagonally dominant or positive definite, see [14, 23]. Its uniqueness and parallelization make it possible to be used in scientific computing, statistics and engineering - see [16, 14, 13, 9, 21] and the references therein. WZ factorization using a parallel computer architecture is known to be faster in computing sparse and dense linear system on $SIMD$ (Single Instruction, Multiple Data) or $MIMD$ (Multiple Instruction, Multiple Data) [7, 22, 17, 2, 1, 12]. The factorization depends on nonsingular central submatrices which execute components in parallel irrespective of the number of processors [10, 11, 18]. Some of the newest forms of WZ factorization, having the properties mentioned above, with potential application in cryptography and graph theory is the WH factorization, see [6, 3, 5]. WH factorization possesses an algorithm which is slightly different from its counterpart WZ factorization by restricting the output entries (specifically the first and last row of the submatrices) to be nonzero. This allows the output matrix (called hourglass matrix) to perfectly resemble an hourglass device, see Figure 1.

WH factorization ($B = WH$) produces W-matrix and H-matrix (hourglass matrix). WH factorization to yield H-matrix can fail to exist even though if the matrix is nonsingular provided the submatrices of the nonsingular matrix are invertible together with all the elements in the first row and in the last row of its submatrices are nonzero, after applying row-interchange. In general, algorithm of WZ factorization is less strict in the output showing that if a nonsingular matrix exhibits WH factorization then WZ factorization is also applicable to the matrix but not otherwise. Throughout the sections, we will assume that matrix B has even order (the assumption is also true for odd order). Then some results on the existence of WH factorization for every strict dominant diagonal matrix, and the block triangular matrices are established.

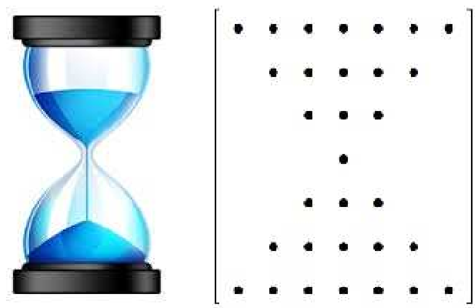

Figure 1: Structural comparison between hourglass device and hourglass matrix.

2 Preliminary and background

To establish some new results on hourglass matrix some terms associated to our findings are included in this section.

Definition 2.1. *[8] A strict dominant diagonal matrix is a square matrix where the element in the diagonal entry in a row is greater than the sum of the elements of all non-diagonal entries in that row. That is*

$$|b_{i,i}| > \sum_{j=1, j\neq i}^{n} |b_{i,j}|. \tag{2.2}$$

Theorem 2.2. *[20] Factorization Theorem Let $B \in R^{n\times n}$ be a nonsingular matrix with a unique QIF factorization, then $B = WZ$ provided that the submatrices of B are invertible.*

Definition 2.3. *[6] An hourglass matrix (H-matrix) is a nonsingular matrix with nonzero entries in the ith and $(n-i+1)$th row of the square matrix of order $n \times n(n \geq 3)$, otherwise 0's, for $i = 1, 2, ..., \lfloor \frac{n+1}{2} \rfloor$. That is,*

$$H = \begin{cases} h_{ij}, & 1 \leq i \leq \lfloor \frac{(n+1)}{2} \rfloor \quad i \leq j \leq n+1-i; \\ h_{ij}, & \lceil \frac{(n+2)}{2} \rceil \leq i \leq n \quad n+1-i \leq j \leq i; \\ 0, & otherwise. \end{cases} \tag{2.3}$$

Now, from Equation (2.3) we can partition H-matrix of even order n into triangular blocks matrices (non-zero entries) as

$$H = \begin{bmatrix} \alpha_{1,1} & \cdots & \cdots & \alpha_{1,\frac{n}{2}} & \beta_{1,\frac{n}{2}+1} & \cdots & \cdots & \beta_{1,n} \\ & \ddots & H_{1,1} & \vdots & \vdots & H_{1,2} & \iddots & \\ & & \ddots & \vdots & \vdots & \iddots & & \\ & & & \alpha_{\frac{n}{2},\frac{n}{2}} & \beta_{\frac{n}{2},\frac{n}{2}+1} & & & \\ & & & \gamma_{\frac{n}{2}+1,\frac{n}{2}} & \delta_{\frac{n}{2}+1,\frac{n}{2}+1} & & & \\ & & \iddots & \vdots & \vdots & \ddots & & \\ & \iddots & H_{2,1} & \vdots & \vdots & H_{2,2} & \ddots & \\ \gamma_{n,1} & \cdots & \cdots & \gamma_{n,\frac{n}{2}} & \delta_{n,\frac{n}{2}+1} & \cdots & \cdots & \delta_{n,n} \end{bmatrix} \tag{2.4}$$

α block, β block, δ block and γ block.

Definition 2.4. *[3] H_{system} is the grouping of H-matrix of order n $(n \geq 4)$ into 2×2 block triangular matrices with each block containing $\lfloor \frac{n}{2} \rfloor \times \lfloor \frac{n}{2} \rfloor$ matrices.*

H_{system} gives four blocks of triangular matrices whenever the dimension (n) of H-matrix is even, such that

$$H_{system} = \begin{bmatrix} H_{1,1} & H_{1,2} \\ H_{2,1} & H_{2,2} \end{bmatrix} \tag{2.5}$$

where

$$H_{1,1} = \begin{cases} h_{ij}, & 1 \leq i \leq \lceil \frac{n-1}{2} \rceil, \;\; i \leq j \leq \lceil \frac{n-1}{2} \rceil; \\ 0, & otherwise. \end{cases}$$

$$H_{1,2} = \begin{cases} h_{ij}, & 1 \leq i \leq \lceil \frac{n-1}{2} \rceil, \;\; \lfloor \frac{n+3}{2} \rfloor \leq j \leq n+1-i; \\ 0, & otherwise. \end{cases}$$

$$H_{2,1} = \begin{cases} h_{ij}, & \lfloor \frac{n+3}{2} \rfloor \leq i \leq n, \;\; \lfloor \frac{n+3}{2} \rfloor \leq j \leq i; \\ 0, & otherwise. \end{cases}$$

$$H_{2,2} = \begin{cases} h_{ij}, & \lfloor \frac{n+3}{2} \rfloor \leq i \leq n, \;\; n+1-i \leq j \leq \lceil \frac{n-1}{2} \rceil; \\ 0, & otherwise. \end{cases}$$

Definition 2.5. *[3] Schur complement of a block matrix are functions of its blocks such that if $H_{1,1}$ (see Equation (2.5)) is invertible then $H_{1,1}$ in H_{system} is*

$$H_{system}/H_{1,1} = H_{2,2} - H_{2,1}H_{1,1}^{-1}H_{1,2}. \tag{2.6}$$

Theorem 2.6. *[3] Schur complement exists in H_{system} only if H-matrix is nonsingular.*

3 SOME RESULTS ON HOURGLASS MATRIX

It should be noted that some results on Z-matrix have been established which are similar to the results obtained here, see for instance [8]. However, the results for hourglass matrix are not directly applicable to Z-matrix.

Theorem 3.1. *If there exists WH factorization for a nonsingular matrix, then there exists WZ factorization.*

Proof. If $B = WH$, then the central submatrices $\Delta_b = b_{i,j}$ of B has the least condition number adopting any matrix norm which are nonsingular according to its factorization algorithm otherwise the factorization fails, for $k = 1, 2, ..., \lfloor \frac{n-1}{2} \rfloor$. If $b_{i,j} = h_{i,j}$ then

$$\Delta_h = \begin{bmatrix} h_{k,k} & \cdots & h_{k,n-k+1} \\ \vdots & & \vdots \\ h_{n-k+1,k} & \cdots & h_{n-k+1,n-k+1} \end{bmatrix}_{1 \le k \le \lfloor \frac{n-1}{2} \rfloor}.$$

This assumption is also applicable to $B = WZ$ according to Theorem 2.2, if and only if its counterpart central submatrices Δz are invertible such that

$$\Delta z = \begin{bmatrix} z_{k,k} & \cdots & z_{k,n-k+1} \\ \vdots & & \vdots \\ z_{n-k+1,k} & \cdots & z_{n-k+1,n-k+1} \end{bmatrix}_{1 \le k \le \lfloor \frac{n-1}{2} \rfloor}.$$

If a nonsingular matrix B assumes WH factorization such that $det\,(\Delta_h) = h_{n-k+1,n-k+1}h_{k,k} - h_{n-k+1,k}h_{k,n-k+1} \neq 0$, then the matrix also assumes WZ factorization such that $det\,(\Delta z) = z_{n-k+1,n-k+1}z_{k,k} - z_{n-k+1,k}z_{k,n-k+1} \neq 0$. However, the computed entry $z_{i,j}$ may or may not be nonzero for $i, j = k, k+1, ..., n-k+1$. This is because WZ factorization only requires invertibility of Δz, whereas WH factorization ensures where necessary that row interchange exists for Δ_h to contain only nonzero entries and still being invertible. In a case where $z_{i,j} \neq 0$ then $z_{i,j} = h_{i,j}$, but if an entry in $z_{i,j}$ is zero then $z_{i,j} \neq h_{i,j}$ even though $det\,(\Delta z) \neq 0$ and $det\,(\Delta_h) \neq 0$. Thus, every WH factorization always implies WZ factorization but the converse is not true. $\square$

Theorem 3.2. *WH factorization exists for every strict dominant diagonal matrix B.*

Proof. Let the matrix B be a strictly dominant diagonal matrix, then

$$|b_{i,i}| > \sum_{j=1, j\neq i}^{n} |b_{i,j}|.$$

The initial step of the factorization is to consider that the entries in the first row are nonzero ($b_{1,j} \neq 0$) and that the entries in the last row are nonzero ($b_{n,j} \neq 0$) for $j = 1, 2, ..., n$. Though all steps of the WH factorization are analogous, we will consider only the first step of factorization.

$$b_{i,j}^{(1)} = b_{i,j} - \frac{b_{i,1}b_{n,n} - b_{i,n}b_{n,1}}{b_{1,1}b_{n,n} - b_{1,n}b_{n,1}} b_{1,j} - \frac{b_{1,1}b_{i,n} - b_{1,n}b_{i,1}}{b_{1,1}b_{n,n} - b_{1,n}b_{n,1}} b_{n,j} = b_{i,j} + u_j b_{i,n} + v_j b_{i,1}$$

where

$$u_j = \frac{b_{1,j}b_{n,1} - b_{n,j}b_{1,1}}{b_{1,1}b_{n,n} - b_{1,n}b_{n,1}} \qquad \text{and} \qquad v_j = \frac{b_{n,j}b_{1,n} - b_{1,j}b_{n,n}}{b_{1,1}b_{n,n} - b_{1,n}b_{n,1}}.$$

Since

$$\sum_{j=2}^{n-1} |u_j| \leq 1 \qquad \text{and} \qquad \sum_{j=2}^{n-1} |v_j| \leq 1$$

$$b_{i,i}^{(1)} = b_{i,i} + u_i b_{i,n} + v_1 b_{i,1}. \tag{3.7}$$

By inequality

$$\begin{aligned} |u_i b_{i,n} + v_1 b_{i,1}| &\leq |u_i||b_{i,n}| + |v_1||b_{i,1}| \\ &\leq |b_{i,n}| + |b_{i,1}| \end{aligned} \tag{3.8}$$

$$-|u_i b_{i,n} + v_1 b_{i,1}| \geq -|b_{i,n}| - |b_{i,1}| \tag{3.9}$$

$$|b_{i,i}| > \sum_{j=1}^{n} |b_{i,j}| = |b_{i,n}| + |b_{i,1}| + \sum_{j=2}^{n-1} |b_{i,j}|. \tag{3.10}$$

Adding Equation (3.9) and Equation (3.10) to obtain

$$|b_{i,i}| - |u_i b_{i,n} + v_1 b_{i,1}| > \sum_{j=2}^{n-1} |b_{i,j}|. \tag{3.11}$$

Since $b_{i,i}^{(1)} = h_{i,i}^{(1)}$ and $b_{k,k} = h_{k,k} \neq 0$ which permits the use of WH factorization, for $k = 1, 2, ..., \frac{n}{2}$. Then, based on Equation (3.7), we can deduce that

$$\begin{aligned} |h_{i,i}^{(1)}| &\geq |h_{i,i}| - |u_i h_{i,n} + v_1 h_{i,1}| \\ &> \sum_{j=2}^{n-1} |h_{i,j}| > 0. \end{aligned}$$

$\square$

Proposition 3.3. *The determinant of H-matrix (of even order) is*

$$\prod_{k=1}^{\frac{n}{2}} (\alpha_{k,k}\delta_{l,l} - \beta_{k,l}\gamma_{l,k})_{l=n-k+1}$$

where $k = 1, 2, ..., \frac{n}{2}$; $l = n - k + 1$.

Proof: Using cofactor expansion to compute the determinant of H-matrix through the sum of minors, we obtain $(n-1) \times (n-1)$ matrices and then expand it along the column.

$$det(H) = det \begin{bmatrix} \alpha_{k,1} & \cdots & \cdots & \alpha_{k,\frac{n}{2}} & \beta_{k,\frac{n}{2}+1} & \cdots & \cdots & \beta_{k,n} \\ & \ddots & \cdots & \vdots & \vdots & \cdots & \iddots & \\ & & \ddots & \vdots & \vdots & \iddots & & \\ & & & \alpha_{k,k} & \beta_{k,l} & & & \\ & & & \gamma_{l,k} & \delta_{l,l} & & & \\ & & \iddots & \vdots & \vdots & \ddots & & \\ & \iddots & \cdots & \vdots & \vdots & \cdots & \ddots & \\ \gamma_{n,k} & \cdots & \cdots & \gamma_{n,\frac{n}{2}} & \delta_{n,\frac{n}{2}+1} & \cdots & \cdots & \delta_{n,n} \end{bmatrix}_{k=1,2,...,\frac{n}{2};\, l=n,n-1,...,n-k+1}$$

$$= (\alpha_{1,1}\delta_{n,n} - \beta_{1,n}\gamma_{n,1}) \cdot det \begin{bmatrix} \alpha_{k,2} & \cdots & \alpha_{k,\frac{n}{2}+1} & \beta_{k,\frac{n}{2}+2} & \cdots & \beta_{k,n+1} \\ & \ddots & \vdots & \vdots & \iddots & \\ & & \alpha_{k,k+1} & \beta_{k,l+1} & & \\ & & \gamma_{l,k+1} & \delta_{l,l+1} & & \\ & \iddots & \vdots & \vdots & \ddots & \\ \gamma_{n,k+1} & \cdots & \gamma_{n,\frac{n}{2}+1} & \delta_{n,\frac{n}{2}+2} & \cdots & \delta_{n,n+1} \end{bmatrix}_{k=2,...,\frac{n}{2}-1;\, l=n-1,...,n-k}$$

$$= (\alpha_{1,1}\delta_{n,n} - \beta_{1,n}\gamma_{n,1}) \cdot (\alpha_{2,2}\delta_{n-1,n-1} - \beta_{2,n-1}\gamma_{n-1,2}) \cdot ... \cdot det \begin{bmatrix} \alpha_{k,k} & \beta_{k,l} \\ \gamma_{l,k} & \delta_{l,l} \end{bmatrix}_{k=\frac{n}{2};\, l=n-k+1}$$

$$= (\alpha_{1,1}\delta_{n,n} - \beta_{1,n}\gamma_{n,1}) \cdot (\alpha_{2,2}\delta_{n-1,n-1} - \beta_{2,n-1}\gamma_{n-1,2}) \cdot ... \cdot \left(\alpha_{\frac{n}{2},\frac{n}{2}}\delta_{n-k+1,n-k+1} - \beta_{\frac{n}{2},n-k+1}\gamma_{n-k+1,\frac{n}{2}}\right)$$

Since $det(H) \neq 0$ then

$$\alpha_{\frac{n}{2},\frac{n}{2}}\delta_{n-k+1,n-k+1} \neq \beta_{\frac{n}{2},n-k+1}\gamma_{n-k+1,\frac{n}{2}}$$

$\forall\ k = 1, 2, ..., \frac{n}{2}$; $l = n - k + 1$. Therefore,

$$(\alpha_{1,1}\delta_{n,n} - \beta_{1,n}\gamma_{n,1}) \cdot (\alpha_{2,2}\delta_{n-1,n-1} - \beta_{2,n-1}\gamma_{n-1,2}) \cdot ... \cdot \left(\alpha_{\frac{n}{2},\frac{n}{2}}\delta_{n-k+1,n-k+1} - \beta_{\frac{n}{2},n-k+1}\gamma_{n-k+1,\frac{n}{2}}\right) \neq 0.$$

$\square$

Corollary 3.4. *If* $\alpha_{k,k}\delta_{l,l} \neq \beta_{k,l}\gamma_{l,k}$ *then* H*-matrix is nonsingular, for* $k = 1, 2, ..., \frac{n}{2}$; $l = n - k + 1$.

Proof. This proof is obvious from Proposition 1, since $\alpha_{k,k}\delta_{l,l} - \beta_{k,l}\gamma_{l,k} \neq 0$. □

Theorem 3.5. *The matrix* $H_{2,2} - H_{2,1}H_{1,1}^{-1}H_{1,2}$ *is a lower triangular invertible matrix if* H_{system} *and* $H_{1,1}$ *are invertible.*

Proof. We have that H-matrix when divided into 4 square $\frac{n}{2} \times \frac{n}{2}$ blocks (that is 2×2 triangular block matrices) gives

$$H_{system} = \begin{bmatrix} H_{1,1} & H_{1,2} \\ H_{2,1} & H_{2,2} \end{bmatrix}. \tag{3.12}$$

Then the Schur complement of the block $H_{1,1}$ on H_{system} is defined in Definition 3. Since

$$H_{1,1} = \begin{bmatrix} \alpha_{k,1} & \cdots & \cdots & \alpha_{k,\frac{n}{2}} \\ 0 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \alpha_{k,k} \end{bmatrix} \tag{3.13}$$

and $det(H_{1,1}) = \alpha_{k,1} \cdot ... \cdot \alpha_{k,k} \neq 0$, $\forall\ k = 1, 2, ..., \frac{n}{2}$. Then, there exists an inverse of the matrix $H_{1,1}$ of the form

$$H_{1,1}^{-1} = \begin{bmatrix} \frac{1}{\alpha_{k,1}} & \cdots & \cdots & * \\ 0 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{1}{\alpha_{k,k}} \end{bmatrix} \tag{3.14}$$

such that

$$H_{2,1}H_{1,1}^{-1} = \begin{bmatrix} 0 & \cdots & 0 & \frac{\gamma_{l,k}}{\alpha_{k,k}} \\ \vdots & \iddots & \iddots & \vdots \\ 0 & \iddots & \vdots & \vdots \\ \frac{\gamma_{n,k}}{\alpha_{k,1}} & \cdots & \cdots & * \end{bmatrix}. \tag{3.15}$$

Thus, the product $H_{2,1}H_{1,1}^{-1}H_{1,2}$ is a lower triangular matrix of the form

$$H_{2,1}H_{1,1}^{-1}H_{1,2} = \begin{bmatrix} \frac{\gamma_{l,k}\beta_{k,l}}{\alpha_{k,k}} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ * & \cdots & \cdots & \frac{\gamma_{n,k}\beta_{k,n}}{\alpha_{k,1}} \end{bmatrix}. \tag{3.16}$$

Therefore,

$$H_{2,2} - H_{2,1}H_{1,1}^{-1}H_{1,2} = \begin{bmatrix} \delta_{l,l} - \frac{\gamma_{l,k}\beta_{k,l}}{\alpha_{k,k}} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ * & \cdots & \cdots & \delta_{n,n} - \frac{\gamma_{n,k}\beta_{k,n}}{\alpha_{k,1}} \end{bmatrix} \quad (3.17)$$

$$det(H_{2,2} - H_{2,1}H_{1,1}^{-1}H_{1,2}) = \left(\delta_{l,l} - \frac{\gamma_{l,k}\beta_{k,l}}{\alpha_{k,k}}\right) \cdot ... \cdot \left(\delta_{n,n} - \frac{\gamma_{n,k}\beta_{k,n}}{\alpha_{k,1}}\right). \quad (3.18)$$

Based on the property of Schur complement, the determinant of Equation (3.18) is nonzero since $H_{2,2} - H_{2,1}H_{1,1}^{-1}H_{1,2}$ is a lower triangular invertible matrix and

$$\frac{det(H_{2,2} - H_{2,1}H_{1,1}^{-1}H_{1,2})}{det(H_{1,1})} \neq 0.$$

It implies that

$$det(H_{2,2} - H_{2,1}H_{1,1}^{-1}H_{1,2}) = \frac{det(H_{system})}{det(H_{1,1})} \neq 0.$$

□

4 Conclusion

It has been shown that WH factorization implies WZ factorization but the converse is not always true. The WH factorization is suitable for factorizing strict dominant diagonal matrix. H-matrix is a nonsingular matrix and its H_{system} has a lower triangular invertible matrix which is always invertible. Hence there exists block WH factorization of a nonsingular matrix.

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