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Matrix Representation of Martingale Measures with Quantized Marginals

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Abstract

A stochastic process $\{S_t\}_{t\in[0,T]}$ is called a martingale process if $\mathbb{E}[S_n|S_m] = S_m$ almost surely for all $m \leq n$. Martingale processes are often used in arbitrage-free pricing of financial assets due to its "fair game" property, that is, when an asset is modeled using martingales then no one can consistently make or lose money through trades in that asset.

In this paper, we focus our attention on random variables that follow a Pareto distribution. We then look at the set of martingale measures on $\mathbb{R}^2$ having marginal measures which are quantized using the so-called U_n-quantization. We shall then represent the transition kernel of such martingale measures as bistochastic matrices. Finally, we shall give some characterization of the set of such matrices.

1 Introduction

A stochastic process $\mathbb{S} = \{S_t\}_{t\in[0,T]}$ is called a *martingale* (process) if it satisfies the condition $\mathbb{E}_{\mathbb{P}}[S_{t_2}|S_{t_1}] = S_{t_1}$, almost surely, for all $t_1 \leq t_2$. In mathematical finance and economics, martingales are often used in pricing

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models. In modeling an arbitrage-free financial asset, one normally uses a martingale or some random process that can be transformed into a martingale via a change of measure. This new measure is referred to as a *risk-neutral measure*. Martingales are vital in non-arbitrage pricing since the martingale property of an asset is equivalent to not being able to get or create arbitrage through trades in that asset. As usually said for martingales, the expected value for tomorrow equals the actual value today.

Given a sequence of random variables $\{S_i : i = 1, 2, \ldots, n\}$, an n-dimensional measure γ is called a *martingale measure* if $\{S_i : i = 1, 2, \ldots, n\}$ satisfy the martingale condition under the expectation with respect to γ. The study on the set of martingale measures first came up as an application of the Theory of Optimal Transportation [7]. From then on, the authors in [1], [2] have applied it to finance with the assumption that the first marginal measure be absolutely continuous with respect to the Lebesgue measure. Furthermore, such papers are focused on the existence and characterization of the martingale measure that optimizes their desired pay-off function. The novelty of this paper is that no one else seems to be looking at properties of the whole set of martingale measures. In this paper, we shall provide a sufficient and necessary condition for convex ordering of two random variables that follow the Pareto distribution. We then apply a discrete approximation to the said distributions, where we shall use the U_n-quantization, proposed by Baker [3]. We then proceed to represent a martingale measure with quantized marginals by a bistochastic matrix. Lastly, we shall give some characterization for the set of such representations .

2 Preliminaries

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and X be a (real-valued) random variable on the said space. This random variable X induces a new probability space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathbb{P}_X)$, with $\mathbb{P}_X(B) = \mathbb{P}(X^{-1}(B))$ for all $B \in \mathcal{B}(\mathbb{R})$. The probability measure $\mathbb{P}_X$ is called the *law* or *distribution* of X. We shall instead use $\mathrm{law}(X)$ to denote the law of X instead of $\mathbb{P}_X$. The function $F_X : \mathbb{R} \to [0, 1]$ such that $F_X(x) = \mathbb{P}_X((-\infty, x])$ for all $x \in \mathbb{R}$ is called the *distribution function* of the random variable X and of $\mathbb{P}_X$.

2.1 Martingale Measures

Let X and Y be random variables over the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$. If Z is a bivariate random variable with joint distribution function given by $F_Z(x, y) = \mathbb{P}(X \leq x, Y \leq y)$, then law$(Z)$ is called a *transport plan* between law(X) and law(Y). Note that law(Z) is a probability measure on $\mathbb{R}^2$, and that the marginal measures of law(Z) are law(X) and law(Y). The set of transport plans between two distributions μ and ν is denoted by $\Pi(\mu, \nu)$.

Example 2.1. *Consider* $X \sim \mathcal{N}(\mu_1, \sigma_1^2)$ *and* $Y \sim \mathcal{N}(\mu_2, \sigma_2^2)$. *If we take a random random vector* $Z = (X, Y) \sim \mathcal{N}\left(\begin{bmatrix}\mu_1 \\ \mu_2\end{bmatrix}, \begin{bmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2\end{bmatrix}\right)$, *with* $\rho \in (-1, 1)$, *then law*(Z), *which is a probability measure on* $\mathbb{R}^2$, *is a transport plan between law*(X) *and law*(Y).

We are interested on a specific subset of $\Pi(\mu, \nu)$. If law$(X) = \mu$ and law$(Y) = \nu$, we then define the set $\mathcal{M}(\mu, \nu)$ to be a subset of $\Pi(\mu, \nu)$ such that any $\pi \in \mathcal{M}(\mu, \nu)$ satisfies $\mathbb{E}_\pi[Y|X] = X$, μ-almost surely. Such measure π is a called a *martingale measure* between μ and ν.

Example 2.2. *Let* $\Omega_1 = \{0, 1\}$. *Take* $\Omega = \Omega_1 \times \Omega_1$, $\mathcal{A} = 2^\Omega$, *and* $\mathbb{P}$ *to be the uniform (discrete) measure on* Ω. *Consider the random variables* X *and* Y *on* $(\Omega, \mathcal{A}, \mathbb{P})$ *with the following distributions: law*$(X) = 0.5(\delta_1 + \delta_3)$ *and law*$(Y) = 0.25(\delta_0 + \delta_2) + 0.5\delta_1$, *where* δ_a *is the Dirac measure centered at* $x = a$. *Suppose we take* $Z = (X, Y) \sim Unif\{(1, 0), (1, 2), (3, 2), (3, 4)\}$. *One can then verify that* $\mathbb{E}[Y|X] = X$ *and hence law*(Z) *is a martingale measure between law*(X) *and law*(Y).

2.2 Convex Ordering

Given two probability measures μ and ν, the set $\Pi(\mu, \nu)$ is always non-empty since product measure $\mu \times \nu$ is always in $\Pi(\mu, \nu)$. However, the set $\mathcal{M}(\mu, \nu)$ may be empty just like in the case where the two random variables have different means. As it turns out, having an additional assumption on the marginal measures leads to the existence of a martingale measure.

Let μ, ν be probability measures on $\mathbb{R}$. We say that μ is dominated by ν in (stochastic) *convex order*, denoted by $\mu \preceq_c \nu$, if for all convex function φ,

$$\mathbb{E}_\mu[\varphi] \leq \mathbb{E}_\nu[\varphi].$$

Moreover, two random variables X and Y are also said to be in convex order, denoted by $X \preceq_c Y$, if $\operatorname{law}(X) \preceq_c \operatorname{law}(Y)$.

If two random variables X and Y are in convex order then by taking $\varphi_1(x) = x$ and $\varphi_2(x) = -x$, it can be shown that $\mathbb{E}[X] = \mathbb{E}[Y]$. Furthermore, using $\varphi_3(x) = x^2$, one will obtain that $\operatorname{Var}(X) \leq \operatorname{Var}(Y)$.
The next theorem shows that convex ordering in the distributions of the marginal random variables is a sufficient and necessary condition for the existence of a martingale measure.

Theorem 2.3 (Strassen [5]). *Let μ, ν be probability measures on $\mathbb{R}$. Then the set $\mathcal{M}(\mu, \nu)$ is non-empty if and only if $\mu \preceq_c \nu$.*

The following lemma, in conjunction with Strassen's Theorem, will be used to prove the sufficient condition for the existence of martingale measures.

Lemma 2.4 (Ohlin [4]). *Suppose X and Y are random variables with finite and equal means. Let F and G be the distribution functions of X and Y, respectively. Then $X \preceq_c Y$ whenever there exists $x_0 \in \mathbb{R}$ such that*

$$F(x) \leq G(x),\ \forall x \leq x_0 \quad \textit{and} \quad F(x) \geq G(x),\ \forall x \geq x_0.$$

Given a random variable X and its distribution function F, we define the *quantile function* of X as

$$F^{-1}(u) = \inf\{x \in \mathbb{R} : F(x) \geq u\}.$$

In the case of a continuous random variable with strictly monotone distribution function, the quantile function is just the usual inverse of the distribution function.

From here on, we only focus on martingale measures over $\mathbb{R}^2$. Even with this streamlining, working with the set of martingale measures with arbitrary marginal measures over $\mathbb{R}$ still proves to be difficult, so we first approximate the marginal measures before looking at the set of martingale measures.

2.3 Quantization of Measure

Quantization is the mapping of possibly infinite values into a finite set. One important application of quantization is on the approximation of a continuous probability measure by a discrete measure with finite support. For this paper, we shall be using the so-called U_n-quantization from [3]. It is called U_n-quantization due to the fact that it produces a uniformly discrete random variable with at most n mass points.

Let X be a random variable with distribution μ and distribution function F. Given $n \in \mathbb{N}$, the U_n-*quantization* of X is the discrete random variable with distribution μ_n, which is a uniform discrete measure with mass points $a_1, a_2, \ldots a_n$, where

$$a_i = n \int_{\frac{i-1}{n}}^{\frac{i}{n}} F^{-1}(u)\, du. \tag{2.1}$$

Here μ_n is referred as the U_n-quantized version of μ.

Example 2.5. *Consider an exponential random variable with $F(x) = 1 - e^{-0.125x}$, $x \geq 0$. Hence the quantile function is given by $F^{-1}(u) = -8\ln(1-u)$, $u \in [0, 1]$. If we take $n = 5$, then the mass points by (2.1) are*

- $a_1 = -8[(-4)\ln(4/5) - 1]$
- $a_2 = -8[(-3)\ln(3/5) + (4)\ln(4/5) - 1]$
- $a_3 = -8[(-2)\ln(2/5) + (3)\ln(3/5) - 1]$
- $a_4 = -8[(-1)\ln(1/5) + (2)\ln(2/5) - 1]$
- $a_5 = -8[\ln(1/5) - 1]$

Note that using this type of quantization doesn't guarantee that the a_i's are unique, but if the original random variable or measure is continuous then the resulting mass points are unique and are in increasing order.

The next theorem shows that this type of approximation is an ideal tool for the purpose of maintaining the existence of martingale measures even after the quantization.

Theorem 2.6 ([3]). *The U_n-quantization preserves convex ordering, that is, if $X \preceq_c Y$ with U and V to be the U_n-quantization of X and Y, respectively, then $U \preceq_c V$.*

3 Matrix Representation

After applying the U_n-quantization to any measure, we can then represent the quantized measure as an n-dimensional vector. It would also be of help, if there is an easy way to represent a martingale measure with quantized marginals.

If Q is a probability measure on $\mathbb{R}^2$, then for any $A, B \in \mathcal{B}(\mathbb{R})$,

$$Q(A \times B) = \sum_{a \in A} \mathbb{P}(Y \in B | X = a)\, \mathbb{P}(X = a).$$

In particular, if a_i and b_j are mass points of the quantized versions of X and Y, respectively, then

$$\begin{aligned} \frac{1}{n} = \mathbb{P}(Y = b_j) &= \sum_{i=1}^{n} \mathbb{P}(X = a_i, Y = b_j) = \sum_{i=1}^{n} \mathbb{P}(Y = b_j | X = a_i)\mathbb{P}(X = a_i) \\ &= \sum_{i=1}^{n} \mathbb{P}(Y = b_j | X = a_i)\frac{1}{n} \end{aligned}$$

which implies that $\sum_{i=1}^{n} \mathbb{P}(Y = b_j | X = a_i) = 1$. Using the same line of reasoning, we get $\sum_{j=1}^{n} \mathbb{P}(Y = b_j | X = a_i) = 1$.

Since we are to fix the marginal measures to be discrete uniform distributions, knowledge of the transport plan is equivalent to the knowledge of its transition probability. Moreover, similar to what is usually done with Markov chains, we can associate every transport plan with a bistochastic matrix $B = (b_{ij})$ that represents the transition probabilities, where $b_{ij} = \mathbb{P}(Y = b_j | X = a_i)$.

A non-negative matrix $B = (b_{ij}) \in M_n(\mathbb{R})$ is said to be an $n \times n$ *bistochastic matrix* if

$$\sum_{k=1}^{n} b_{ik} = 1 \quad \text{and} \quad \sum_{k=1}^{n} b_{kj} = 1, \text{ for all } i, j = 1, 2, \ldots, n$$

One known result regarding bistochatic matrices is from Birkhoff.

Theorem 3.1 (Birkhoff). *The set of all $n \times n$ bistochastic matrices, denoted by D_n, is an $(n-1)^2$-dimensional polytope in $\mathbb{R}^{n^2}$.*

4 Results

We now present our results starting with the sufficient and necessary condition for convex ordering of two Pareto distributed random variables, followed by the quantization of the desired marginal measures. We then give a characterization, similar to that of Birkhoff, to the set of bistochastic matrices that represent a martingale measure with quantized marginals. Lastly, we look at certain sets of 3×3 bistochastic matrices.

4.1 Convex Ordering

A random variable X is said to follow a *Pareto Distribution* with shape parameter $a > 0$ and scale parameter $k > 0$, denoted by $X \sim \text{Pareto}(a, k)$, if

$$F(x) = 1 - \left(\frac{k}{x}\right)^a \mathbb{1}_{[k,\infty)}(x).$$

Theorem 4.1. *If* $X \sim$ *Pareto*(a, k), *then* $\mathbb{E}[X] = \dfrac{ak}{a-1}$, $a > 1$ *and Var*$(X) = \dfrac{ak^2}{(a-1)^2(a-2)}$, $a > 2$.

The next result gives us a necessary and sufficient condition for the existence of martingale measures with marginal measures that follow a Pareto distribution.

Proposition 4.2. *Suppose* $X \sim$ *Pareto*(α, k) *and* $Y \sim$ *Pareto*(β, j) *with* $\alpha, \beta > 2$. *Then,* $X \preceq_c Y$ *if and only if* $\alpha \geq \beta$, $k \geq j$, *and* $\dfrac{\alpha k}{\alpha - 1} = \dfrac{\beta j}{\beta - 1}$.

Proof. If $X \preceq_c Y$ then $\mathbb{E}[X] = \mathbb{E}[Y]$, and hence $\dfrac{\alpha k}{\alpha - 1} = \dfrac{\beta j}{\beta - 1}$. Moreover $\text{Var}(X) \leq \text{Var}(Y)$, which then implies that $\dfrac{\beta}{\alpha - 2} \leq \dfrac{\alpha}{\beta - 2}$. From this inequality, it can then be shown that $\alpha \geq \beta$ and eventually $k \geq j$.
Conversely, if we take the distribution functions of X and Y as $F_X(x) = 1 - \left(\dfrac{k}{x}\right)^\alpha$ and $F_Y(y) = 1 - \left(\dfrac{j}{y}\right)^\beta$, and assume that $\alpha \geq \beta$, $k \geq j$, and $\dfrac{\alpha k}{\alpha - 1} = \dfrac{\beta j}{\beta - 1}$, then $F_X \leq F_Y$ for all $x \leq \left(\dfrac{k^\alpha}{j^\beta}\right)^{\frac{1}{\alpha-\beta}}$ and $F_X \geq F_Y$ for all $x \geq \left(\dfrac{k^\alpha}{j^\beta}\right)^{\frac{1}{\alpha-\beta}}$. So, by Lemma 2.4, $X \preceq_c Y$. $\square$

4.2 Quantized Version of the Pareto Distribution

Proposition 4.3. *Suppose $X \sim Pareto(\alpha, k)$, then its U_n-quantized version has mass points given by*

$$a_i = n^{\frac{1}{\alpha}} k \frac{\alpha}{\alpha - 1} \left[(n - i + 1)^{\frac{\alpha-1}{\alpha}} - (n - i)^{\frac{\alpha-1}{\alpha}} \right] \quad i = 1, \dots, n. \tag{4.2}$$

The formula for the a_i's is a direct consequence of (2.1) and the fact that $F_X(x) = 1 - \left(\frac{k}{x}\right)^{\alpha}$, with $x > k$, while its quantile function is given by $F_X^{-1}(u) = k(1-u)^{\frac{-1}{\alpha}}$, with $u \in (0, 1)$.

4.3 Matrix Representation

We now present a characterization result for the set of martingale measures with U_n-quantized marginals. This theorem works for any marginal distribution as long as the U_n-quantization gives a support of n distinct points.

Proposition 4.4. *Let X and Y be distinct, continuous random variables such that $law(X) = \mu$, $law(Y) = \nu$ and $X \preceq_c Y$. Let $n \in \mathbb{N}$, and take μ_n and ν_n to be the U_n-quantized version of μ and ν, respectively. Then, the set of transition kernels corresponding to the elements of $\mathcal{M}(\mu_n, \nu_n)$ is an $(n-1)(n-2)$-dimensional polytope.*

Proof. Let μ_n have mass points $a_1, \dots, a_n$ while ν_n have mass points $b_1, \dots, b_n$, obtained by using formula (2.1). Note that these mass points are arranged in increasing order. If $Q \in \mathcal{M}(\mu_n, \nu_n)$, then there exists an $n \times n$ bistochastic matrix $M = (m_{ij})$ corresponding to Q such that $m_{ij} = \mathbb{P}[Y = b_j | X = a_i]$. We then have the following equations satisfied by the entries of M.

$$\sum_{k=1}^{n} m_{ik} = 1, \qquad i = 1, 2, \dots, n \qquad [Equations\ (1) - (n)]$$

$$\sum_{k=1}^{n} m_{kj} = 1, \qquad j = 1, 2, \dots, n \qquad [Equations\ (n+1) - (2n)]$$

$$\sum_{k=1}^{n} b_k m_{ik} = a_i, \qquad i = 1, 2, \dots, n \qquad [Equations\ (2n+1) - (3n)]$$

The first $2n$ equations are needed so that M is a bistochastic matrix, while the last set of n equations guarantees that Q is a martingale measure. It can be shown that equation $(2n)$ is a linear combination of equations (1) to $(2n-1)$. Moreover, equation $(2n+1)$ can also be shown as a linear combination of the remaining $3n-2$ equations. Lastly, the remaining $3n-2$ equations can be verified to be linearly independent. Hence, by using the rank-nullity theorem, the dimension of the solution space is $n^2-(3n-2)=(n-2)(n-1)$. Due to the fact that all the entries of M are between 0 and 1, the resulting set $\mathcal{M}(\mu_n,\nu_n)$ will be the intersection of the above solution space and of all flat regions having the form $\{e \in \mathbb{R}^{n^2} : e(i) \in [0,1], \text{ for some } i \leq n^2\}$. $\square$

Lastly, we look at the case when $n=3$ and that the marginals follow a Pareto distribution.

Proposition 4.5. *Let $X \sim$ Pareto(α,k) and $Y \sim$ Pareto(β,j) be Pareto-distributed random variables such that law$(X)=\mu$ and law$(Y)=\nu$ and $X \preceq_c Y$. If μ_3 and ν_3 are the U_3-quantization of μ and ν, respectively, then any martingale measure in $\mathcal{M}(\mu_3,\nu_3)$ can be associated to a matrix of the form*

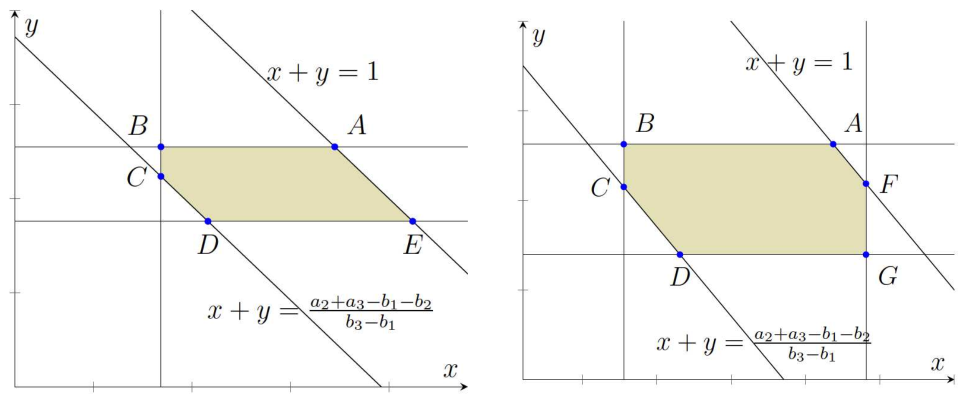
$$\begin{bmatrix} \frac{b_3-b_2}{b_1-b_2}x+\frac{b_3-b_2}{b_1-b_2}y+\frac{b_1+b_2-a_2-a_3}{b_1-b_2} & \frac{b_3-b_1}{b_2-b_1}x+\frac{b_3-b_1}{b_2-b_1}y+\frac{b_1+b_2-a_2-a_3}{b_2-b_1} & 1-x-y \\ \frac{a_2-b_2}{b_1-b_2}-\frac{b_3-b_2}{b_1-b_2}x & \frac{a_2-b_1}{b_2-b_1}-\frac{b_3-b_1}{b_2-b_1}x & x \\ \frac{a_3-b_2}{b_1-b_2}-\frac{b_3-b_2}{b_1-b_2}y & \frac{a_3-b_1}{b_2-b_1}-\frac{b_3-b_1}{b_2-b_1}y & y \end{bmatrix},$$

where $a_i = 3^{\frac{1}{\alpha}}k\frac{\alpha}{\alpha-1}\left[(3-i+1)^{\frac{\alpha-1}{\alpha}}-(3-i)^{\frac{\alpha-1}{\alpha}}\right]$
and $\quad b_i = 3^{\frac{1}{\beta}}j\frac{\beta}{\beta-1}\left[(3-i+1)^{\frac{\beta-1}{\beta}}-(3-i)^{\frac{\beta-1}{\beta}}\right]$, *for all* $i=1,2,3$.
The point (x,y) would lie on the following feasible regions depending on three cases:

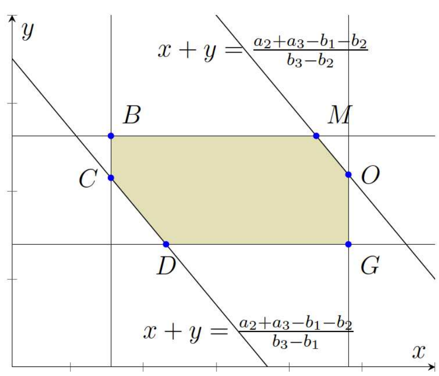
Case 1: $b_3-a_3 \leq a_2-b_1$ *and* $\dfrac{b_3-a_3}{b_3-b_2} \leq \dfrac{a_2-b_1}{b_3-b_1}$.

Case 2: $b_3-a_3 \leq a_2-b_1$ *and* $\dfrac{b_3-a_3}{b_3-b_2} > \dfrac{a_2-b_1}{b_3-b_1}$.

Case 3: $b_3-a_3 > a_2-b_1$.



Feasible regions of Case 1 and Case 2



Feasible regions of Case 3

where

$$A = \left(\frac{b_3-a_3}{b_3-b_1}, \frac{a_3-b_1}{b_3-b_1}\right), \quad C = \left(\frac{a_2-b_2}{b_3-b_2}, \frac{a_3b_3+a_2b_1-a_2b_2-a_3b_2-b_1b_3+b_2^2}{(b_3-b_1)(b_3-b_2)}\right),$$

$$B = \left(\frac{a_2-b_2}{b_3-b_2}, \frac{a_3-b_1}{b_3-b_1}\right), \quad D = \left(\frac{a_2b_3+a_3b_1-a_2b_2-a_3b_2-b_1b_3+b_2^2}{(b_3-b_1)(b_3-b_2)}, \frac{a_3-b_2}{b_3-b_2}\right),$$

$$E = \left(\frac{b_3-a_3}{b_3-b_2}, \frac{a_3-b_2}{b_3-b_2}\right), \quad M = \left(\frac{a_3b_3+a_2b_3-b_2b_3-a_2b_1-a_3b_1+b_1^2}{(b_3-b_1)(b_3-b_2)}, \frac{a_3-b_1}{b_3-b_1}\right),$$

$$F = \left(\frac{a_2-b_1}{b_3-b_1}, \frac{b_3-a_2}{b_3-b_1}\right), \quad O = \left(\frac{a_2-b_1}{b_3-b_1}, \frac{a_3b_3+a_2b_2-b_2b_3-a_2b_1-a_3b_1+b_1^2}{(b_3-b_1)(b_3-b_2)}\right).$$

$$G = \left(\frac{a_2-b_1}{b_3-b_1}, \frac{a_3-b_2}{b_3-b_2}\right),$$

Proof. Since $n = 3$, by Proposition 4.4, we expect that the desired set of 3×3 bistochastic matrices has dimension 2. Furthermore, by using the algorithm stated in the proof of the Proposition 4.4 and by taking m_{23} and m_{33} as the generators, one would then yield the desired matrix. The corresponding cases and their respective feasible regions arose from the restriction that the matrix entries should be between 0 and 1.

□

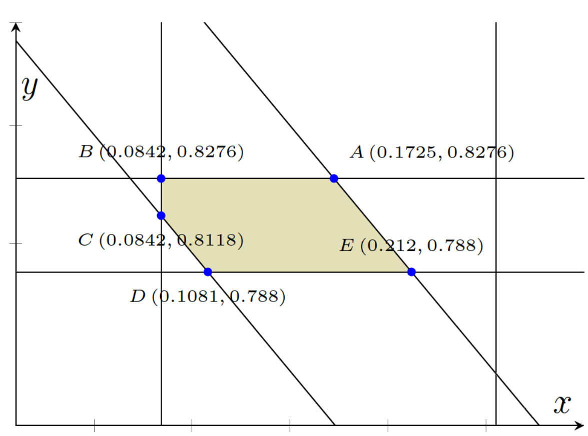
It should be noted that the matrix representation given in the previous proposition will work for any pair of continuous marginal distributions as long as the two are in convex order, with the a_i's and b_j's to be the mass points obtained after quantization.

Example 4.6. *Let $X \sim$ Pareto$(4,3)$ and $Y \sim$ Pareto$\left(3, \frac{8}{3}\right)$. Using Proposition 4.2, we get $X \preceq_c Y$. Taking $n = 3$, the mass points of the U_3-quantized versions of X and Y are given below*

i	*1*	*2*	*3*
a_i	$3^{\frac{1}{4}}(4)\left[3^{\frac{3}{4}} - 2^{\frac{3}{4}}\right]$	$3^{\frac{1}{4}}(4)\left[2^{\frac{3}{4}} - 1\right]$	$3^{\frac{1}{4}}(4)$
b_i	$3^{\frac{1}{3}}(4)\left[3^{\frac{2}{3}} - 2^{\frac{2}{3}}\right]$	$3^{\frac{1}{3}}(4)\left[2^{\frac{2}{3}} - 1\right]$	$3^{\frac{1}{3}}(4)$

Then, based on Proposition 4.5, any bistochastic matrix that has the form seen below, with (x, y) lying on the following region, is a transition matrix of some martingale measure.

$$\begin{bmatrix} -4.3561x - 4.3561y + 4.7992 & 5.3561x + 5.3561y - 4.7992 & 1 - x - y \\ -0.3668 + 4.3561x & 1.3668 - 5.3561x & x \\ -3.4324 + 4.3561y & 4.4324 - 5.3561y & y \end{bmatrix}$$



5 Summary and Further Works

Using the U_n-quantization on Pareto distributed random variables, we came up with discrete marginal measures having n support points. Furthermore, since this type of quantization preserves the convex ordering of the original measures, we came up with a set of martingale measures which are then represented by bistochastic matrices. It was also shown that using such quantization, the set of martingale measures is $(n-2)(n-1)$-dimensional. Future direction for this kind of research can involve the use of a different quantization technique as well as the characterization of the set of martingale measures having continuous marginal measures (non-quantized). Also of interest would be the use of such sets in optimizing certain reward/cost functions and examining if convergence is achieved as the number of support points increase.

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