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The Performance of Multivariate Exponentially Weighted Moving Average Control Chart Based on Regression Adjustment when the Multivariate Normality Assumption is Violated

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Abstract

The violation of multivariate normality assumption affects multivariate control chart in many ways especially the Multivariate Exponentially Weighted Moving Average (MEWMA) control chart which has been designed to have faster detection capability. Regression adjustment has been developed as a scheme that consists of plotting univariate control charts of the residuals from each variable obtained when that variable is regressed on all the others and the Average Run Length (ARL) performance of this scheme is very competitive. In this paper, we investigate the ARL performance of the MEWMA control chart with and without the use of regression adjusted variables when the multivariate normality assumption is violated. The

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MEWMA based on regression adjustment provides a larger out-of-control ARL values than the MEWMA control chart.

1 Introduction

The MEWMA control chart is one of various types of the multivariate control charts that has been discussed widely. It uses weighted averages of previously observed random vectors to monitor the mean vector of the process. A quality characteristic of a manufactured product that is measured on a numerical scale is called a *variable*. Suppose we have p random variables (quality characteristics) given by $X_1, X_2, \ldots, X_p$. Write these random variables in terms of a random vector $\boldsymbol{X} = (X_1, X_2, ..., X_p)'$. If $X_1, X_2, \ldots, X_p$ are independent normal random variables with mean μ_i and variance σ_i^2 for each variable X_i, then the random vector $\boldsymbol{X}$ has an independent multivariate normal distribution and is denoted by $\boldsymbol{X} \sim N_p(\boldsymbol{\mu}, \sigma^2\boldsymbol{I})$ with mean vector $E(\boldsymbol{X}) = \boldsymbol{\mu}$ and covariance matrix $V(\boldsymbol{X}) = \sigma^2\boldsymbol{I}$, respectively.

For the MEWMA control chart, a multivariate normal distribution is an important assumption that is used to describe the behavior of quality characteristic of interest. However, in many real situations, the assumption is not always met. Stoumbos and Sullivan [1] displayed the robustness to non-normality of the MEWMA control chart and they also used multivariate t and multivariate gamma distributions and compared the performance of control chart to multivariate normal process data. Nidsunkid et al. [2] studied the effects of violations of the multivariate normality assumption in multivariate Shewhart and MEWMA control charts when the random vector ($\boldsymbol{X}$) is from the multivariate normal, multivariate t, multivariate uniform, multivariate beta and multivariate lognormal distributions. Nidsunkid et al. [3] studied the performance of the control chart for MCUSUM control charts when the multivariate normality assumption is violated. In addition, the impact of a random vector with variables from normal and non-normal distributions on multivariate control charts was proposed by Nidsunkid et al. [4]. The type of skewness for a distribution, even sampling data, are from a finite population [5] and all affect the performance control chart in different ways.

The control chart based on regression adjusted variables has been developed by Hawkins [6]. The scheme essentially consists of plotting univariate control charts of the residuals from each variable obtained when that variable is regressed on all the others. Residual control charts are very applicable to individual measurements which occur frequently in practice with multivariate data. Hawkins showed that the ARL performance of this scheme is very competitive with other methods, but depends on the types of control charts applied to the residuals.

In this research, we examine the performance of MEWMA and MEWMA based on regression adjustment control charts when the multivariate normality assumption is violated. We compute the statistical performance of the control chart and report it in terms of the ARL.

2 The Multivariate Exponentially Weighted Moving Average Control Chart

The MEWMA control chart is a good alternative to the multivariate Shewhart control chart to improve the detection of small shifts in the mean vector. The MEWMA control chart uses weighted averages of previously observed random vectors to monitor the mean vector of the process. Lowry et al. [7] have developed a MEWMA control chart. Let $\boldsymbol{X}_i$, $i = 1, 2, \ldots$ be a $p \times 1$ random vector that follows a p-variate normal distribution. The MEWMA is defined as follows

$$\boldsymbol{Z}_i = \lambda \boldsymbol{X}_i + (1-\lambda)\boldsymbol{Z}_{i-1}, \tag{2.1}$$

where $0 \leq \lambda \leq 1$ is the weighting constant and $\boldsymbol{Z}_0$ is equal to the in-control mean vector of the process. The quantity for sample i plotted on the control chart is

$$T_i^2 = \boldsymbol{Z}_i' \boldsymbol{\Sigma}_{\boldsymbol{Z}_i}^{-1} \boldsymbol{Z}_i, \tag{2.2}$$

where the covariance matrix is

$$\boldsymbol{\Sigma}_{\boldsymbol{Z}_i} = \frac{\lambda}{2-\lambda}[1-(1-\lambda)^{2i}]\boldsymbol{\Sigma}. \tag{2.3}$$

The MEWMA control chart gives an out-of-control signal if $T_i^2 > H$, where $H > 0$ denotes an upper control limit, and is calculated (by simulation) for the process to achieve a specified in-control ARL. An approximation of the covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{Z}_i}$ as i approaches infinity is as follows

$$\boldsymbol{\Sigma}_{\boldsymbol{Z}_i} = \frac{\lambda}{2-\lambda}\boldsymbol{\Sigma}. \tag{2.4}$$

Lowry et al. [7] suggested that the ARL performance of the MEWMA chart depends on the mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ only through the value of the noncentrality parameter δ, where

$$\delta = (\boldsymbol{\mu}' \boldsymbol{\Sigma} \boldsymbol{\mu})^{1/2}. \tag{2.5}$$

Basically, large values of δ correspond to bigger shifts in the mean. The value $\delta = 0$ represents the in-control state. Note that, for a given shift size, ARLs generally tend to increase as λ increases, except for very large values of δ (or large shifts). Since the MEWMA with $\lambda = 1$ is equivalent to the multivariate Shewhart control chart, the MEWMA with $\lambda < 1$ is more sensitive to a smaller shift in the mean vector [8] .

3 Regression Adjustment

Hawkins [6], [9] introduced the regression adjustment based on $\mathbf{Z}$ scale. To find the vector of scaled residuals, let

$$\mathbf{Y} = \mathbf{\Sigma}^{-1}(\boldsymbol{X} - \boldsymbol{\mu}_0). \tag{3.6}$$

The ith component of $\mathbf{Y}$ is the regression residual when variable i is regressed on all other variables, scaled by a factor π_{ii}^{-1}, where π_{ij}^{-1} is the i, jth element of $\mathbf{\Sigma}^{-1}$. Under control, $\mathbf{Y} \sim N(\mathbf{0}, \mathbf{\Sigma}^{-1})$. $\mathbf{Z}$ is just a rescaling of $\mathbf{Y}$:

$$\mathbf{Z} = [diag(\mathbf{\Sigma}^{-1})]^{-1/2}\mathbf{Y} = \mathbf{A}(\boldsymbol{X} - \boldsymbol{\mu}_0), \tag{3.7}$$

where the transformation matrix

$$\mathbf{A} = [diag(\mathbf{\Sigma}^{-1})]^{-1/2}\mathbf{\Sigma}^{-1}. \tag{3.8}$$

By definition, it follows that $\mathbf{Z} \sim N(\mathbf{0}, \mathbf{B})$, where

$$\mathbf{B} = [diag(\mathbf{\Sigma}^{-1})]^{-1/2}\mathbf{\Sigma}^{-1}[diag(\mathbf{\Sigma}^{-1})]^{-1/2}. \tag{3.9}$$

The $\boldsymbol{X}$ and $\mathbf{Z}$ scale correspond to the Hotelling T^2 statistic. The Hotelling T^2 statistic is

$$T^2 = (\boldsymbol{X} - \boldsymbol{\mu}_0)'\mathbf{\Sigma}^{-1}(\boldsymbol{X} - \boldsymbol{\mu}_0). \tag{3.10}$$

The T^2 can be expressed as

$$T^2 = \sum_{i=1}^{p}(X_i - \mu_i)Y_i, \tag{3.11}$$

or, rewriting in terms of $\mathbf{Z}$,

$$T^2 = \sum_{i=1}^{p} W_i, \tag{3.12}$$

where $W_i = (X_i - \mu_i)Z_i\pi_{ii}^{-1/2}$.

The residuals obtained from regression technique are plotted on the MEWMA charts to monitor the shift in means separately since the residuals are considered as independent and approximately normally distributed.

4 Methodology and Simulation Study

For this research, MEWMA control chart and MEWMA based on regression adjustment control chart which monitor the mean vector of the process are considered. For each chart, we investigate the average run length (ARL) when the distribution of quality characteristic (variable) in random vector is normal (N), t, uniform (U), beta (B), and lognormal (L) for 3, 5 and 10 process variables. The specific distributions include several symmetric (normal, t, uniform), left-skewed (beta) and right-skewed (lognormal) distributions. The control limit was chosen to provide an in-control ARL of 200.

5 Results

As a random vector with variables from normal and non-normal distributions, a comparison of the MEWMA and MEWMA based on regression adjustment control charts are shown in Tables 1-3. The out-of-control ARLs vary inversely with percentage of shifts in mean vector. For MVN distributions, the in-control ARLs are approximately the same expected value, 200, and the out-of-control ARLs for regression adjustment are larger than MEWMA. The in-control ARLs for N-t and N-L regression adjustment are less than MEWMA, while the out-of-control ARLs with large size of shift are greater than MEWMA. The N-U and N-B have similar characeristics, the in-control and out-of-control ARLs for regression adjustment are greater than MEWMA ARLs.

6 Conclusions

When multivariate normality assumption is violated, the performance of MEWMA control chart based on regression adjustment was compared with MEWMA control chart. Because of the large value of out-of-control ARLs, MEWMA based on regression adjustment detected changes in the mean later than MEWMA, especially when the random vector contains uniform or beta distributions.

Table 1: The ARLs performance of MEWMA control chart and MEWMA based on regression adjustment control chart for 3 variables when multivariate normality assumption is violated

MEWMA						
λ	Distributions	Shifts in mean vector				
		0	5%	10%	15%	20%
0.1	N-N-N	200.05	112.01	47.63	25.09	16.14
	N-t-N	184.87	109.10	47.62	25.24	16.26
	N-U-N	211.98	114.78	47.67	25.17	16.09
	N-B-N	204.61	116.54	48.19	25.17	16.19
	N-L-N	182.79	99.96	45.77	25.09	16.24
0.2	N-N-N	199.66	133.89	62.68	31.24	18.28
	N-t-N	163.61	117.59	60.28	31.13	18.43
	N-U-N	237.60	150.86	65.81	31.64	18.26
	N-B-N	210.23	147.72	67.07	32.23	18.45
	N-L-N	152.22	100.03	53.92	29.56	17.94
0.4	N-N-N	199.69	157.62	91.24	49.20	27.77
	N-t-N	128.78	110.43	74.20	44.87	26.89
	N-U-N	281.24	211.33	113.08	56.00	29.62
	N-B-N	223.68	192.15	109.92	56.66	30.15
	N-L-N	111.71	86.80	58.62	38.19	24.03
MEWMA based on regression adjustment						
λ	Distributions	Shifts in mean vector				
		0	5%	10%	15%	20%
0.1	N-N-N	200.21	186.07	139.50	84.97	49.95
	N-t-N	94.02	91.39	76.92	54.82	36.56
	N-U-N	334.77	309.33	218.90	125.16	70.37
	N-B-N	255.47	245.89	179.50	105.67	60.32
	N-L-N	75.95	73.05	63.00	46.93	32.40
0.2	N-N-N	200.37	187.03	139.32	85.17	50.24
	N-t-N	94.43	91.56	76.82	54.53	36.38
	N-U-N	333.63	311.96	220.12	125.75	70.50
	N-B-N	255.04	247.94	179.47	105.30	60.39
	N-L-N	75.99	73.12	62.75	47.23	32.21
0.4	N-N-N	200.01	188.50	139.92	85.29	50.39
	N-t-N	94.44	91.37	76.45	54.66	36.34
	N-U-N	333.39	310.47	219.05	125.83	71.00
	N-B-N	255.79	247.74	180.17	105.19	60.07
	N-L-N	76.40	72.57	62.99	47.02	32.42

Table 2: The ARLs performance of MEWMA control chart and MEWMA based on regression adjustment control chart for 5 variables when multivariate normality assumption is violated

MEWMA						
λ	Distributions	Shifts in mean vector				
		0	5%	10%	15%	20%
0.1	N-N-N-N-N	199.99	98.90	38.00	19.77	12.97
	N-t-N-t-N	179.01	95.10	37.70	19.80	12.98
	N-U-N-U-N	216.14	101.84	38.08	19.74	12.90
	N-B-N-B-N	204.23	103.66	38.33	19.80	12.95
	N-L-N-L-N	171.83	85.80	36.48	19.71	12.94
0.2	N-N-N-N-N	200.12	123.14	50.54	23.64	13.69
	N-t-N-t-N	150.18	103.33	47.90	23.37	13.68
	N-U-N-U-N	246.21	140.75	53.01	23.94	13.67
	N-B-N-B-N	211.31	138.56	54.28	24.16	13.72
	N-L-N-L-N	133.37	83.75	42.11	22.25	13.49
0.4	N-N-N-N-N	200.24	150.42	78.44	38.33	20.08
	N-t-N-t-N	109.67	92.83	59.94	34.12	19.44
	N-U-N-U-N	305.71	216.23	100.10	43.55	21.27
	N-B-N-B-N	228.64	190.39	97.30	44.08	21.81
	N-L-N-L-N	89.26	69.03	45.70	28.13	17.32
MEWMA based on regression adjustment						
λ	Distributions	Shifts in mean vector				
		0	5%	10%	15%	20%
0.1	N-N-N-N-N	199.65	196.67	186.34	165.23	130.75
	N-t-N-t-N	74.86	74.37	72.45	68.66	60.76
	N-U-N-U-N	406.72	395.75	370.34	320.68	245.92
	N-B-N-B-N	272.05	280.25	276.28	249.41	196.16
	N-L-N-L-N	56.71	56.09	54.45	51.78	46.82
0.2	N-N-N-N-N	200.14	196.55	186.99	165.01	131.70
	N-t-N-t-N	74.47	73.88	72.72	68.74	61.02
	N-U-N-U-N	405.00	399.73	373.19	321.33	246.42
	N-B-N-B-N	273.15	281.51	277.03	251.20	197.79
	N-L-N-L-N	57.15	55.99	54.58	51.86	46.82
0.4	N-N-N-N-N	200.20	196.75	186.92	164.91	131.60
	N-t-N-t-N	74.53	73.93	72.72	68.75	61.08
	N-U-N-U-N	405.03	399.56	373.24	321.62	246.50
	N-B-N-B-N	273.28	281.64	277.19	251.12	197.78
	N-L-N-L-N	57.05	56.15	54.64	51.85	46.89

Table 3: The ARLs performance of MEWMA control chart and MEWMA based on regression adjustment control chart for 10 variables when multivariate normality assumption is violated

MEWMA						
λ	Distributions	Shifts in mean vector				
		0	5%	10%	15%	20%
0.1	N-N-N-N-N-N-N-N-N-N	200.29	80.76	27.39	14.58	9.92
	N-t-N-t-N-N-t-N-t-N	174.84	77.67	27.34	14.62	9.92
	N-U-N-U-N-N-U-N-U-N	215.90	83.33	27.36	14.62	9.90
	N-B-N-B-N-N-B-N-B-N	204.58	83.56	27.66	14.61	9.89
	N-L-N-L-N-N-L-N-L-N	164.17	70.16	26.72	14.63	9.95
0.2	N-N-N-N-N-N-N-N-N-N	200.40	106.42	35.61	15.83	9.44
	N-t-N-t-N-N-t-N-t-N	141.49	87.74	34.14	15.79	9.40
	N-U-N-U-N-N-U-N-U-N	245.53	119.69	37.17	16.04	9.41
	N-B-N-B-N-N-B-N-B-N	214.05	118.17	37.74	16.18	9.45
	N-L-N-L-N-N-L-N-L-N	122.11	70.11	30.76	15.31	9.39
0.4	N-N-N-N-N-N-N-N-N-N	199.94	138.37	60.21	25.29	12.46
	N-t-N-t-N-N-t-N-t-N	97.87	79.52	45.94	23.12	12.16
	N-U-N-U-N-N-U-N-U-N	311.37	197.99	74.52	27.79	12.91
	N-B-N-B-N-N-B-N-B-N	231.19	176.07	73.39	28.15	13.00
	N-L-N-L-N-N-L-N-L-N	77.38	57.83	35.30	19.77	11.33
MEWMA based on regression adjustment						
λ	Distributions	Shifts in mean vector				
		0	5%	10%	15%	20%
0.1	N-N-N-N-N-N-N-N-N-N	200.32	196.91	186.80	174.02	156.53
	N-t-N-t-N-N-t-N-t-N	63.19	62.60	61.87	60.34	58.34
	N-U-N-U-N-N-U-N-U-N	448.95	438.90	407.02	366.45	320.02
	N-B-N-B-N-N-B-N-B-N	276.28	285.14	282.06	270.69	249.82
	N-L-N-L-N-N-L-N-L-N	46.26	46.04	45.37	44.39	43.19
0.2	N-N-N-N-N-N-N-N-N-N	200.14	196.78	187.76	173.09	155.83
	N-t-N-t-N-N-t-N-t-N	63.26	62.64	61.77	60.02	57.84
	N-U-N-U-N-N-U-N-U-N	443.48	435.06	404.76	365.78	319.38
	N-B-N-B-N-N-B-N-B-N	276.23	285.78	283.10	270.41	249.40
	N-L-N-L-N-N-L-N-L-N	46.41	46.14	45.46	44.41	42.71
0.4	N-N-N-N-N-N-N-N-N-N	200.13	197.66	187.71	173.75	156.91
	N-t-N-t-N-N-t-N-t-N	62.95	62.85	61.67	60.29	58.15
	N-U-N-U-N-N-U-N-U-N	445.70	437.39	407.25	364.77	320.83
	N-B-N-B-N-N-B-N-B-N	276.94	285.00	282.70	269.63	250.20
	N-L-N-L-N-N-L-N-L-N	46.53	46.08	45.68	44.18	42.59

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