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Ideal Structure of Kumjian-Pask Algebras

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Abstract

Let R be a unital commutative ring and let Λ be a finitely aligned k-graph which has no cycle with $|\Lambda| < \infty$. Through a purely matrix approach, we show that the ideal structure of R implies the ideal structure of the Kumjian-Pask algebra $KP_R(\Lambda)$.

1 Introduction

Suppose Λ is a k-graph without sources and R is a commutative unital ring R. Pino, Clark, Huef and Raeburn [1] introduced the Kumjian-Pask algebra $KP_R(\Lambda)$, which is a higher-rank analogues of the Leavitt path algebras $L_K(E)$ of directed graph E over a field K [2].

Many researchers were interested to investigate deeper properties of this algebra or to make generalizations; for example, [3, 4, 5, 6, 7, 9, 8, 10]. Recently, Clark and Pangalela [6] generalized the theory of Kumjian-Pask algebra for finitely aligned k-graphs and then proved the graded-invariant uniqueness theorem.

Rosjanuardi and Gozali [10] described the Kumjian-Pask algebra of a finitely aligned k-graph Λ with $|\Lambda| < \infty$ which has no cycle. Theorem 5.2 of

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[11] implies that the Kumjian-Pask algebra $KP_R(\Lambda)$ is nothing but a direct sum of matrix algebras. When Λ has a single pure source [10, Proposition 3.4], an ideal I of R induced a short exact sequence of R-algebras

$$0 \to IKP_R(\Lambda) \to KP_R(\Lambda) \to KP_{R/I}(\Lambda) \to 0.$$

We extend the results in [10] to a more general case. When Λ has many pure sources, we obtain a similar result to [10, Proposition 3.4]. In Theorem 3.4 we prove that if Λ is a finitely aligned k-graph with $|\Lambda| < \infty$ which has no cycle, then an ideal J of R induced a short exact sequence of R-algebras.

2 Preliminaries on Kumjian-Pask Algebras

For any k-graph Λ and $n \in \mathbb{N}^k$ the notation Λ^n stands for the set of paths of degree n and the notation $\Lambda^{\neq 0}$ is reserved for the set of paths of nonzero degree $\{\lambda \in \Lambda : d(\lambda) \neq 0\}$. For $v \in \Lambda^0$ and $n \in \mathbb{N}^k$, the set of degree n paths with range v is denoted by $v\Lambda^n$. The graph Λ is called *row-finite* if for every $v \in \Lambda^0$ and $m \in \mathbb{N}^k$, the set $v\Lambda^m$ is finite. For $v \in \Lambda^0$, we define $v\Lambda := \{\lambda \in \Lambda : r(\lambda) = v\}$ and $\Lambda v := \{\lambda \in \Lambda : s(\lambda) = v\}$. A vertex $v \in \Lambda^0$ is called a *source* if there is $n \in \mathbb{N}^k$ such that $v\Lambda^n = \emptyset$. When there is no edge going to v, we write $\{v\} = v\Lambda$ and this vertex is called a *pure source*. The k-graph Λ is said to *have no sources* if $v\Lambda^m \neq \emptyset$, for all $v \in \Lambda^0$ and $m \in \mathbb{N}^k$.

Let Λ be a k-graph and suppose $\mu, \nu \in \Lambda$. A *minimal common extension* (MCE) of μ and ν is a path $\lambda \in \Lambda$ in which $d(\lambda) = d(\mu) \vee d(\nu)$ and $\lambda(0, d(\mu)) = \mu, \lambda(0, d(\nu)) = \nu$. The set $\mathrm{MCE}(\mu, \nu)$ consists of all of MCEs of μ and ν and we use the notation $\Lambda^{\min}(\mu, \nu) = \{(\alpha, \beta) \in \Lambda \times \Lambda : \mu\alpha = \nu\beta \in \mathrm{MCE}(\mu, \nu)\}$. A k-graph Λ is called *finitely aligned* if $|\Lambda^{\min}(\mu, \nu)| < \infty$, for every $\mu, \nu \in \Lambda$. For $v \in \Lambda^0$, a set $E \subset v\Lambda$ is called *exhaustive* if for every $\mu \in v\Lambda$, there is $\lambda \in E$ such that $\Lambda^{\min}(\lambda, \mu) \neq \emptyset$. We define $\mathrm{FE}(\Lambda) = \bigcup_{v \in \Lambda^0} \{E \subseteq v\Lambda \setminus \{v\} : E \text{ finite and exhaustive}\}$. For any $\lambda \in \Lambda$, the ghost path is denoted by λ^* with degree $d(\lambda^*) = -d(\lambda)$, source $s(\lambda^*) = r(\lambda)$, and range $r(\lambda^*) = s(\lambda)$. The set of all ghost paths is denoted by $G(\Lambda)$ and when the vertices are excluded we denote it by $G(\Lambda^{\neq 0})$. The composition on $G(\Lambda)$ is defined by setting $\lambda^*\mu^* = (\mu\lambda)^*$ for $\lambda, \mu \in \Lambda^{\neq 0}$ with $r(\mu^*) = s(\lambda^*)$.

Definition 2.1. *[6] Suppose Λ is a finitely aligned k-graph and R is a unital commutative ring. A Kumjian-Pask Λ family $\{S_\lambda, S_{\mu^*} : \lambda, \mu \in \Lambda\}$ in an R-algebra A consists of a function $S : \Lambda \cup G(\Lambda^{\neq 0}) \to A$ satisfying*

(KP1) $\{S_v : v \in \Lambda^0\}$ is a family of mutually orthogonal idempotents,

(KP2) for all $\lambda, \mu \in \Lambda$ *with* $s(\lambda) = r(\mu)$, *we have* $S_\lambda S_\mu = S_{\lambda\mu},\ S_{\mu^*} S_{\lambda^*} = S_{(\lambda\mu)^*}$,

(KP3) $S_{\lambda^*} S_\mu = \sum_{(\rho,\tau)\in\Lambda^{\min}(\lambda,\mu)} S_\rho S_{\tau^*}$, *for all* $\lambda, \mu \in \Lambda$,

(KP4) $\prod_{\lambda\in E}(S_{r(E)} - S_\lambda S_{\lambda^*}) = 0$ *for all* $E \in FE(\Lambda)$.

Suppose Λ is a finitely aligned k-graph and R is a unital ring. Theorem 3.7 of [6] assures us that there is an R-algebra $KP_R(\Lambda)$ generated by the Kumjian-Pask family $\{s_\lambda, s_{\mu^*} : \lambda, \mu \in \Lambda\}$ which is universal for the Kumjian-Pask family.

3 Kumjian-Pask algebras of finitely aligned k-graphs with many pure sources

When there are many pure sources, we will be dealing with the direct sum of matrices algebras $\oplus_{v\Lambda=\{v\}} M_{|\Lambda v|}(R)$ as in [11]. Hence the ideal $IKP_R(\Lambda)$ will be also identified as the direct sum $I(\oplus_{v\Lambda=\{v\}} M_{|\Lambda v|}(R))$. For a unital algebra R and natural numbers $1 \le k, l \le n$, the notation ${}_nE_{kl}$ is reserved for the unit matrix in $M_n(R)$. For $n \in \mathbb{N}$, we use the notation ${}_nO_R$ for the zero matrix in the algebra $M_n(R)$.

Lemma 3.1. *Let* R *be a unital commutative ring. If* J *is an ideal of* R, *then* $J(\oplus_{i=1}^n M_{n_i}(R)) = \oplus_{i=1}^n M_{n_i}(J)$ *is an ideal of the* R*-algebra* $\oplus_{i=1}^n M_{n_i}(R)$.

Proof. Any element of $J(\oplus_{i=1}^n M_{n_i}(R))$ can be written as a finite linear combination $\sum_{k\in F} \gamma_k \alpha_k$ where F is a finite subset of $\mathbb{N}$, $\gamma_k \in J$ and $\alpha_k = (\alpha_{k_i})_{i=1}^n \in \oplus_{i=1}^n M_{n_i}(R)$. Therefore, $J(\oplus_{i=1}^n M_{n_i}(R)) \subset \oplus_{i=1}^n M_{n_i}(J))$, because $\gamma_k \alpha_{k_i} \in J$ and $\sum_{k\in F} \gamma_k \alpha_k = \sum_{k\in F} \gamma_k (\alpha_{k_i})_{i=1}^n = \sum_{k\in F} (\gamma_k \alpha_{k_i})_{i=1}^n = (\sum_{k\in F} \gamma_k \alpha_{k_i})_{i=1}^n$.

We write $A_i \in M_{n_i}(J)$ in term of entries as $[a_{i_{kl}}]_i$. Hence, $(A_i)_{i=1}^n \in \oplus_{i=1}^n M_{n_i}(J)$ can be written as $([a_{i_{kl}}]_i)_{i=1}^n$, which is nothing but

$(\sum_{k=1}^{n_i}\sum_{l=1}^{n_i} a_{i_{kl}}({}_{n_i}E_{kl})_i)_{i=1}^{n}$

$$= \sum_{m=1}^{n}(B_{m,i})_{i=1}^{n} \text{ where } B_{m,i} = \begin{cases} \sum_{k=1}^{n_i}\sum_{l=1}^{n_i} a_{i_{kl}}(E_{kl})_i, & \text{if } i = m \\ {}_{n_{m+1}}O_J, & \text{if } i = m+1 \\ \vdots & \vdots \\ {}_{n_n}O_J, & \text{if } i = n, \end{cases}$$

$$= \sum_{m=1}^{n}\sum_{k=1}^{n_i}\sum_{l=1}^{n_i} a_{m_{kl}}(C_{m,i})_{i=1}^{n} \text{ where } C_{m,i} = \begin{cases} ({}_{n_i}E_{kl}), & \text{if } i = m \\ {}_{n_{m+1}}O_R, & \text{if } i = m+1 \\ \vdots & \vdots \\ {}_{n_n}O_R, & \text{if } i = n. \end{cases}$$

Hence $(A_i)_{i=1}^{n} \in \oplus_{i=1}^{n} M_{n_i}(J)$ can be written as a linear combination of elements in $\oplus_{i=1}^{n} M_{n_i}(R)$ with scalars in J which implies that $\oplus_{i=1}^{n} M_{n_i}(J) \subset J(\oplus_{i=1}^{n} M_{n_i}(R))$. Therefore, the equality holds.
Finally, for each $i \in \mathbb{N}$, $M_{n_i}(J)$ is an ideal of the algebra $M_{n_i}(R)$ and hence $\oplus_{i=1}^{n} M_{n_i}(J)$ is an ideal of $\oplus_{i=1}^{n} M_{n_i}(R)$. Consequently, $J(\oplus_{i=1}^{n} M_{n_i}(R))$ is an ideal of $\oplus_{i=1}^{n} M_{n_i}(R)$. □

Theorem 3.2. *Let R be a unital commutative ring. Suppose Λ is a finitely aligned k-graph with $|\Lambda| < \infty$, $\{v_1, v_2, ..., v_n\} \subset \Lambda^0$ such that $v_i\Lambda = \{v_i\}$ and contains no cycle. If J is an ideal of R, then $JKP_R(\Lambda)$ is an ideal of $KP_R(\Lambda)$.*

Proof. Theorem 5.2 of [11] implies that $KP_R(\Lambda) = \oplus_{i=1}^{n} M_{|\Lambda v_i|}(R)$. Therefore, $\oplus_{i=1}^{n} M_{|\Lambda v_i|}(J)$ is an ideal of $\oplus_{i=1}^{n} M_{|\Lambda v_i|}(R)$, which is nothing but $JKP_R(\Lambda)$ by Lemma 3.1. □

Theorem 3.3. *Let R be a unital commutative ring. If J is an ideal of R, then there is a short exact sequence*

$$0 \to \oplus_{i=1}^{n} M_{n_i}(J) \to \oplus_{i=1}^{n} M_{n_i}(R) \to \oplus_{i=1}^{n} M_{n_i}(R/J) \to 0.$$

Proof. Since J and R can be viewed as R-modules, J, R and R/J are R-algebras. Now consider the canonical injection $\psi : J \to R$ and the canonical quotient map $\phi : R \to R/J$. Then $0 \to J \xrightarrow{\psi} R \xrightarrow{\phi} R/J \to 0$. Define $\bar{\psi} : \oplus_{i=1}^{n} M_{n_i}(J) \to \oplus_{i=1}^{n} M_{n_i}(R)$ by $\bar{\psi}(([a_{i_{kl}}]_i)_{i=1}^{n}) = ([\psi(a_{i_{kl}})]_i)_{i=1}^{n}$. Then $\bar{\psi}$ is an R-algebra homomorphism, because ψ is. Injectivity of ψ implies that $\bar{\psi}$ is injective, because

$$\ker \bar{\psi} = \{([a_{i_{kl}}]_i)_{i=1}^{n} \in \oplus_{i=1}^{n} M_{n_i}(J) | [\psi(a_{i_{kl}})]_i = {}_{n_i}O_J \text{ for } i = 1, ..., n\}.$$

Define $\bar{\phi} : \oplus_{i=1}^{n} M_{n_i}(R) \to \oplus_{i=1}^{n} M_{n_i}(R/J)$ by $\bar{\phi}(([b_{i_{kl}}]_i)_{i=1}^{n}) = ([\phi(b_{i_{kl}})]_i)_{i=1}^{n}$. Then $\bar{\phi}$ is a surjective R-algebra homomorphism, because ϕ is. Since

$$\begin{aligned}\ker\bar{\phi} &= \{([a_{i_{kl}}]_i)_{i=1}^{n} \in \oplus_{i=1}^{n} M_{n_i}(R) | a_{i_{kl}} \in \ker\phi = \operatorname{Im}\psi\} \\ &= \{([\psi(b_{i_{kl}})]_i)_{i=1}^{n} \in \oplus_{i=1}^{n} M_{n_i}(R) | b_{i_{kl}} \in J\} \\ &= \{\bar{\psi}(([b_{i_{kl}}]_i)_{i=1}^{n}) \in \oplus_{i=1}^{n} M_{n_i}(R) | b_{i_{kl}} \in J\} = \operatorname{Im}\bar{\psi},\end{aligned}$$

we get the short exact sequence. □

Theorem 3.4. *Let R be a unital commutative ring. Suppose Λ is a finitely aligned k-graph with $|\Lambda| < \infty$ and $\{v_1, v_2, ..., v_n\} \subset \Lambda^0$ such that $v_i\Lambda = \{v_i\}$ and contains no cycle. If J is an ideal of R, then there is a short exact sequence*

$$0 \to JKP_R(\Lambda) \to KP_R(\Lambda) \to KP_{R/J}(\Lambda) \to 0.$$

Proof. Applying [11, Theorem 5.2] to the rings R and R/J we get

$$KP_R(\Lambda) = \oplus_{i=1}^{n} M_{|\Lambda v_i|}(R) \text{ and } KP_{R/J}(\Lambda) = \oplus_{i=1}^{n} M_{|\Lambda v_i|}(R/J).$$

By Lemma 3.1, we get $JKP_R(\Lambda) = J(\oplus_{i=1}^{n} M_{n_i}(R)) = \oplus_{i=1}^{n} M_{n_i}(J)$. Theorem 3.3 implies that $0 \to JKP_R(\Lambda) \to KP_R(\Lambda) \to KP_{R/J}(\Lambda) \to 0$ is a short exact sequence. □

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