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Estimation of the fuzzy parameters for Exponential-Rayleigh Distribution by using a nonlinear Ranking function

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Abstract

Statistical procedures are employed to estimate the parameters of any distribution or model. In this article using a new procedure to find the fuzzy numbers for the estimated parameters of mixture distribution which called Exponential-Rayleigh Distribution (ERD) by applying the fuzzy interval estimation to create a Non Linear-Pentagonal Membership Function (NL-PMF) and its Ranking function after performing and deriving the estimation method (Ordinary Least Squares Estimation Method (OLSEM)) which is provided using Newton-Raphson (NR) technique. Following that finding, we will compared the survival function's estimator before and after fuzzy with

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the actual data via utilizing the mean squared error to show that which one is the best.

1 Introduction

THe Exponential-Rayleigh Distribution (ERD) is one of the mixture distributions for analyzing lifetime data between the cumulative distribution function of the Exponential distribution and the cumulative distribution function of the Rayleigh distribution. In 2021, the authors of [1] introduced this distribution depending on the mixture cumulative distribution function of the Exponential distribution and the Rayleigh distribution. When certainties happen, the persons normally recall the reached data as well as try to forecast the upcoming incidents. The main technique that is normally employed as the theory of probability has satisfied these needs for dealing with uncertainty and inaccuracy. Nevertheless, for the uncertainty circumstances, the theory of probability may not be sufficient and so the integration between statistical logic and fuzzy logic is needed to improve the robustness [2]. The theory of fuzzy sets, discovered by Zadeh [3] in 1965 takes a significant part in the statistical fields. We estimate the parameters of the Exponential-Rayleigh distributions by using the Ordinary Least Squares Estimation Method (OLSEM) depending on the Newton-Raphson procedure. Moreover, we apply the interval estimation method to find the lower and upper intervals to get fuzzy numbers. Furthermore, we propose a Non Linear-Pentagonal Membership Function (NL-PMF) and construct the nonlinear Ranking function to solve these fuzzy numbers. Many researchers [4]-[7] have presented pentagonal fuzzy numbers with various types in order to clear the vagueness of the existing issues. The fuzzy survival (reliability) function collects with fuzzy mathematics. In the analysis of the fuzzy survival function, the factors of the membership function being adopted as a certainty function for obtaining the concrete data of survival [8]. To know the optimum values for these parameters of this distribution, we make a comparison between the survival functions prior and beyond the fuzzy to choose the best from method of lowest mean squared error. The rest of the present paper is organized as follows:

In section 2, we define ERD and mention some properties. In section 3, we discuss the estimation method steps for OLSE. In section 4, we illustrate some basic concepts of the fuzzy set and number. In section 5, we construct the NL-PMF. In section 6, we deal with the ranking function to convert the factors from fuzzy into crisp. In section 7, we describe an application for real data. Finally, in section 8, we present the numerical part which is to compute and compare the results between the traditional survival function and the fuzzy survival function by using the Mean Squared Error (MSE) to show the best one is presented.

2 Exponential-Rayleigh Distribution [1]

In this section, we give some mathematical and statistical properties for ERD.

- **The probability density function (pdf) of ERD is:**

$$f(x;\alpha,\beta) = \begin{cases} (\alpha+\beta x)e^{-\left(\alpha x+\frac{\beta}{2}x^2\right)}, & x \geq 0 \\ 0, & otherwise, \end{cases} \tag{2.1}$$

where $\Omega = \{\alpha, \beta; \alpha > 0, \beta > 0\}$ and α, β are scale parameters.

- **The cumulative distribution function (cdf) of ERD is:**

$$F(x) = \int_0^x f(u)du = 1 - e^{-\left(\alpha x+\frac{\beta}{2}x^2\right)}, \qquad x \geq 0; \alpha, \beta > 0 \tag{2.2}$$

- **The survival function $S(t)$ of ERD is:**

$$S(t;\alpha,\beta) = 1 - F(t) = e^{-\left(\alpha t+\frac{\beta}{2}t^2\right)}, \qquad t \geq 0; \alpha, \beta > 0 \tag{2.3}$$

- **The hazard rate function $h(t)$ of ERD is:**

$$h(t;\alpha,\beta) = \alpha + \beta t, \qquad t \geq 0; \alpha, \beta > 0 \tag{2.4}$$

3 Ordinary Least Squares Estimation Method

This method is based on a linear regression model (straight line equation) by taking the sum squared error as follows:
$y = \beta_0 + \beta_1 x + \epsilon, \epsilon = y - \beta_0 - \beta_1 x$
$\sum_{i=1}^n \epsilon_i^2 = \sum_{i=1}^n (y_i - \hat{y}_i)^2,$
where y_i is the actual value and $\hat{y}_i$ is the predicted value.
By utilizing the cumulative distribution function (cdf) of ERD, we replace y_i with $F(x_i)$ and $\hat{y}_i$ with $\hat{F}(x_i)$ so that:
$\sum_{i=1}^n \epsilon_i^2 = \sum_{i=1}^n \left(F(x_i) - \hat{F}(x_i)\right)^2,$

where $F(x_i) = \frac{i-0.5}{n}$ is the empirical cdf and $\hat{F}(x_i) = 1 - e^{-\left(\alpha x_i + \frac{\beta}{2}x_i^2\right)}$. Therefore,

$$\sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} \left[\frac{i-0.5}{n} - \left(1 - e^{-\left(\alpha x_i + \frac{\beta}{2}x_i^2\right)}\right)\right]^2,$$

or

$$\sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} \left[e^{-\left(\alpha x_i + \frac{\beta}{2}x_i^2\right)} - \frac{n-i+0.5}{n}\right]^2.$$

Using the notation $S(\alpha, \beta)$ for $\sum_{i=1}^{n} \epsilon_i^2$, we have

$$S(\alpha, \beta) = \sum_{i=1}^{n} \left[e^{-\left(\alpha x_i + \frac{\beta}{2}x_i^2\right)} - \frac{n-i+0.5}{n}\right]^2. \tag{3.5}$$

We get the partial derivatives from (3.5) with respect to the parameters α, β as follows:

$$\frac{\partial S(\alpha, \beta)}{\partial \alpha} = 2\sum_{i=1}^{n} \frac{n-i+0.5}{n} x_i e^{-\left(\alpha x_i + \frac{\beta}{2}x_i^2\right)} - 2\sum_{i=1}^{n} x_i e^{-2\left(\alpha x_i + \frac{\beta}{2}x_i^2\right)} \tag{3.6}$$

$$\frac{\partial S(\alpha, \beta)}{\partial \beta} = \sum_{i=1}^{n} \frac{n-i+0.5}{n} x_i^2 e^{-\left(\alpha x_i + \frac{\beta}{2}x_i^2\right)} - \sum_{i=1}^{n} x_i^2 e^{-2\left(\alpha x_i + \frac{\beta}{2}x_i^2\right)} \tag{3.7}$$

Equations (3.6) and (3.7) form a system of non-linear equations that is complicated to solve. Therefore, we let $f(\alpha) = \frac{\partial S(\alpha,\beta)}{\partial \alpha}$, and $f(\beta) = \frac{\partial S(\alpha,\beta)}{\partial \beta}$. Then

$$f(\alpha) = 2\sum_{i=1}^{n} \frac{n-i+0.5}{n} x_i e^{-\left(\alpha x_i + \frac{\beta}{2}x_i^2\right)} - 2\sum_{i=1}^{n} x_i e^{-2\left(\alpha x_i + \frac{\beta}{2}x_i^2\right)} \tag{3.8}$$

$$f(\beta) = \sum_{i=1}^{n} \frac{n-i+0.5}{n} x_i^2 e^{-\left(\alpha x_i + \frac{\beta}{2}x_i^2\right)} - \sum_{i=1}^{n} x_i^2 e^{-2\left(\alpha x_i + \frac{\beta}{2}x_i^2\right)} \tag{3.9}$$

We solve the non-linear system numerically. Newton-Raphson (NR) method is one of the popular methods that is used to solve these equations and find the estimated values of the unknown parameters depending on the Jacobian matrix J. The elements of J are found as follows:

$$\frac{\partial f(\alpha)}{\partial \alpha} = 4\sum_{i=1}^{n} x_i^2 e^{-2\left(\alpha x_i + \frac{\beta}{2}x_i^2\right)} - 2\sum_{i=1}^{n} \frac{n-i+0.5}{n} x_i^2 e^{-\left(\alpha x_i + \frac{\beta}{2}x_i^2\right)} \tag{3.10}$$

$$\frac{\partial f(\alpha)}{\partial \beta} = 2\sum_{i=1}^{n} x_i^3 e^{-2\left(\alpha x_i + \frac{\beta}{2}x_i^2\right)} - \sum_{i=1}^{n} \frac{n-i+0.5}{n} x_i^3 e^{-\left(\alpha x_i + \frac{\beta}{2}x_i^2\right)} \tag{3.11}$$

$$\frac{\partial f(\beta)}{\partial \beta} = \sum_{i=1}^{n} x_i^4 e^{-2\left(\alpha x_i + \frac{\beta}{2} x_i^2\right)} - \frac{1}{2}\sum_{i=1}^{n} \frac{n-i+0.5}{n} x_i^4 e^{-\left(\alpha x_i + \frac{\beta}{2} x_i^2\right)} \tag{3.12}$$

$$\frac{\partial f(\beta)}{\partial \alpha} = 2\sum_{i=1}^{n} x_i^3 e^{-2\left(\alpha x_i + \frac{\beta}{2} x_i^2\right)} - \sum_{i=1}^{n} \frac{n-i+0.5}{n} x_i^3 e^{-\left(\alpha x_i + \frac{\beta}{2} x_i^2\right)} \tag{3.13}$$

The Jacobian matrix J consists of the partial derivatives with respect to the parameters α and β:

$$\begin{bmatrix} \frac{\partial f(\alpha)}{\partial \alpha} & \frac{\partial f(\alpha)}{\partial \beta} \\ \frac{\partial f(\beta)}{\partial \alpha} & \frac{\partial f(\beta)}{\partial \beta} \end{bmatrix}$$

Now, we obtain the estimators for the unknown parameters $\hat{\alpha}$ and $\hat{\beta}$ of ERD from the formula:

$$\begin{bmatrix} \alpha_{k+1} \\ \beta_{k+1} \end{bmatrix} = \begin{bmatrix} \alpha_k \\ \beta_k \end{bmatrix} - J^{-1} \begin{bmatrix} f(\alpha) \\ f(\beta) \end{bmatrix} \quad k = 0, 1, ..., n \tag{3.14}$$

where the initial values α_0 and β_0 will be assumed to get the results assuming the stop error terms for α and β as follows:

$$\begin{bmatrix} \alpha_{k+1} \\ \beta_{k+1} \end{bmatrix} - \begin{bmatrix} \alpha_k \\ \beta_k \end{bmatrix} = \begin{bmatrix} \epsilon_{k+1}(\alpha) \\ \epsilon_{k+1}(\beta) \end{bmatrix}$$

4 Fuzzy set theory [9]

In this section, we introduce some basic concepts of fuzzy set theory.

- Let X be a universal set which is an objects' classical set. We define a fuzzy set $\tilde{A}$ on X as:
  $\tilde{A} = \{(X, \mu_{\tilde{A}}(X)) : x \in X, \mu_{\tilde{A}}(X) : X \to [0,1]\}$, where the map $\mu_{\tilde{A}}(X) : X \to [0,1]$ is called the membership degree of element x in the fuzzy set $\tilde{A}$.

- For a given $\alpha \in [0,1]$, the ordinary set of elements that belong to the fuzzy set $\tilde{A}$ is made up of members whose membership are not less than α is called the α-level set or the α-cut set; that is, $A_\alpha = \{x \in X : \mu_{\tilde{A}}(x) \geq \alpha\}$.

- The fuzzy numbers are very special fuzzy subsets of the real numbers (i.e., a fuzzy no. is a fuzzy set $\tilde{A}$ on the actual line R) such that:

1. there exists at least one point $x_0 \in R, \mu_{\tilde{A}}(x_0) = 1$ (x_0 is called a mean value of $\tilde{A}$).
2. the membership is a piecewise continuous function.
3. a fuzzy set $\tilde{A}$ is normal and convex.

5 Non-Linear Pentagonal Membership Function

The structure of this function is defined as follows:
Let $\tilde{A}_{NL-PMF} = (a, b, c, d, e; r, w)$ be fuzzy numbers with two weight functions r, w. Then

$$M_{\tilde{A}NL-PMF}(X) = \begin{cases} r\left(\frac{x-a}{b-a}\right)^m & a \leq x < b \\ r + (w-r)\left(\frac{x-b}{c-d}\right)^m & b \leq x < c \\ w & x = c \\ r + (w-r)\left(\frac{d-x}{d-c}\right)^m & c \leq x < d \\ r\left(\frac{e-x}{e-d}\right)^m & d \leq x < e \\ 0 & x < c \text{ or } x > e, \end{cases}$$

where $0 < r < w \leq 1$.
Now, by using the λ-cut, $\lambda \in [0, 1]$, we have
- Let $\lambda = r\left(\frac{x-a}{b-a}\right)^m \to x = a + \left(\frac{b-a}{\frac{1}{rm}}\right)\lambda^{\frac{1}{m}} = inf_1\tilde{A}(\lambda)$
- Let $\lambda = r + (w-r)\left(\frac{x-b}{c-b}\right)^m \to x = b + \left(\frac{c-b}{(w-r)^{\frac{1}{m}}}\right)(\lambda - r)^{\frac{1}{m}} = inf_2\tilde{A}(\lambda)$
- Let $\lambda = r + (w-r)\left(\frac{d-x}{d-c}\right)^m \to x = d - \left(\frac{d-c}{(w-r)^{\frac{1}{m}}}\right)(\lambda - r)^{\frac{1}{m}} = sup_1\tilde{A}(\lambda)$
- Let $\lambda = r\left(\frac{e-x}{e-d}\right)^m \to x = e - \left(\frac{e-d}{\frac{1}{rm}}\right)\lambda^{\frac{1}{m}} = sup_2\tilde{A}(\lambda)$

We can define the λ-cut of NL-PMF as:
$A_\lambda = \left\{x \in X : M_{\tilde{A}_{NL-PMF}}(x) \geq \lambda\right\}$,
$A_\lambda =$

$$\begin{cases} \alpha + \left(\frac{b-a}{\frac{1}{rm}}\right)\lambda^{\frac{1}{m}} = inf_1\tilde{A}(\lambda) & on[0, r] \\ b + \frac{(c-b)}{(w-r)^{\frac{1}{m}}}(\lambda - r)^{\frac{1}{m}} = inf_2\tilde{A}(\lambda) & on[r, w] \\ d - \frac{(d-c)}{(w-r)^{\frac{1}{m}}}(\lambda - r)^{\frac{1}{m}} = sup_1\tilde{A}(\lambda) & on[r, w] \\ e - \left(\frac{e-d}{\frac{1}{rm}}\right)\lambda^{\frac{1}{m}} = sup_2\tilde{A}(\lambda) & on[0, r], \end{cases}$$

where, $r, w \in (0,1]$ and $\lambda \in [0,1]$

6 Ranking function

The ranking function $R : F(R) \to \mathbb{R}$ maps every fuzzy number into the real line [10].
Now, we apply the ranking function depending on NL-PMF.
Let $R_{NL}(\tilde{A}) = \frac{1}{2}[I_1 + I_2]$, where
$I_1 = \int_0^r inf_1\tilde{A}(\lambda)d\lambda + \int_r^w inf_2\tilde{A}(\lambda)d\lambda$, $I_2 = \int_r^w sup_1\tilde{A}(\lambda)d\lambda + \int_r^w sup_2\tilde{A}(\lambda)d\lambda$
$I_1 = \int_0^r [a + \left(\frac{b-a}{\frac{1}{rm}}\right)\lambda^{\frac{1}{m}}]d\lambda + \int_r^w [b + \frac{(c-b)}{(w-r)^{\frac{1}{m}}}(\lambda - r)^{\frac{1}{m}}]d\lambda$
$I_1 = \left[a\lambda + \left(\frac{b-a}{\frac{1}{rm}}\right)\frac{\lambda^{\frac{1}{m}+1}}{\frac{1}{m}+1}\right]\Big|_0^r + \left[b\lambda + \frac{(c-b)}{(w-r)^{\frac{1}{m}}}\frac{(\lambda-r)^{\frac{1}{m}+1}}{\frac{1}{m}+1}\right]\Big|_r^w$
$I_1 = \left[ar + \left(\frac{b-a}{\frac{1}{rm}}\right)\frac{r^{\frac{1}{m}+1}}{\frac{1}{m}+1}\right] + \left[bw + \frac{(c-b)}{(w-r)^{\frac{1}{m}}}\frac{(w-r)^{\frac{1}{m}+1}}{\frac{1}{m}+1} - br - \frac{(c-b)}{(w-r)^{\frac{1}{m}}}\frac{(r-r)^{\frac{1}{m}+1}}{\frac{1}{m}+1}\right]$
$I_1 = \left[ar = \frac{m(b-a)}{m+1}r\right] + \left[b(w-r) + \frac{m(c-b)}{m+1}(w-r)\right]\left[\frac{am+a+bm+an}{m+1}\right](r)$
$+\left[\frac{bm+b+cm-bm}{m+1}\right](w-r)$
$I_1 = \left[\frac{a+bm}{m+1}\right](r) + \left[\frac{b+cm}{m+1}\right](w-r)$
$I_2 = \int_r^w d - \frac{(d-c)}{(w-r)^{\frac{1}{m}}}(\lambda - r)^{\frac{1}{m}}d\lambda + \int_0^r e - \frac{(e-d)}{\frac{1}{rm}}\lambda^{\frac{1}{m}}d\lambda$
$I_2 = \left[d\lambda - \frac{(d-c)}{(w-r)^{\frac{1}{m}}}\frac{(\lambda-r)^{\frac{1}{m}+1}}{\frac{1}{m}+1}\right]\Big|_r^w \left[e\lambda - \frac{(e-d)}{\frac{1}{rm}}\frac{\lambda^{\frac{1}{m}+1}}{\frac{1}{m}+1}\right]\Big|_0^r$
$I_2 = \left[dw - \frac{(d-c)}{(w-r)^{\frac{1}{m}}}\frac{(w-r)^{\frac{1}{m}+1}}{\frac{1}{m}+1} - dr + \frac{(d-c)}{(w-r)^{\frac{1}{m}}}\frac{(r-r)^{\frac{1}{m}+1}}{\frac{1}{m}+1}\right] + \left[er - \frac{(e-d)}{\frac{1}{rm}}\frac{r^{\frac{1}{m}+1}}{\frac{1}{m}+1}\right]$
$I_2 = \left[d(w-r) - \frac{m(d-c)}{n+1}(w-r)\right] + \left[er - \frac{m(e-d)}{m+1}r\right] = \left[\frac{dm+d+cm-dm}{m+1}\right](w-r)$
$+\left[\frac{em+e+dm-em}{m+1}\right](r)$
$I_2 = \left[\frac{cm+d}{m+1}\right](w-r) + \left[\frac{dm+e}{m+1}\right](r)$
$R_{NL}(\tilde{A}) = \frac{1}{2}[\left[\frac{a+bm}{m+1}\right](r) + \left[\frac{b+cm}{m+1}\right](w-r) + \left[\frac{cm+d}{m+1}\right](w-r) + \left[\frac{dm+e}{m+1}\right](r)]$

$$R_{NL}(\tilde{A}) = \frac{1}{2(m+1)}[(a + m(b+d) + e)(r) + (b + 2mc + d)(w-r)] \quad (6.15)$$

7 Application

As an application of our work, we collected real data from Yarmouk Teaching Hospital in Iraq for patients with a myocardial infarction (MI), commonly known as a heart attack for the period from 1/1/2022 to 31/10/2022. MI is

one of the common and dangerous diseases that threaten the life of individuals and may lead to sudden death. Note that the number of patients during this period is 468 where 414 patients survived and 54 patients died:
$T = \{1, 2, 1, 2, 2, 5, 1, 1, 2, 4, 9, 1, 6, 3, 3, 3, 11, 2, 2, 2, 4, 4, 2, 1, 2, 2, 9, 1, 3, 1, 5, 4, 2, 9, 2, 11, 2, 2, 3, 2, 2, 2, 4, 3, 3, 3, 1, 3, 4, 2, 3, 2, 7, 1\}$.

8 Numerical Results

In this part, we compute and estimate the unknown parameters for ERD by the following steps:

1. After using the OLSEM to estimate two parameters for ERD, implement the Newton-Raphson method in MATLAB programming under complete data.

2. Select different initial values (α_0, β_0) to find the numerical values for the estimated parameters $(\hat{\alpha}, \hat{\beta})$ from equation (3.14), then finding the survival function $\hat{S}(t)$ with these values.

3. Apply the confidence interval estimation (CIE) for the points estimation $(\hat{\alpha}, \hat{\beta})$ to limit the lower and upper of interval as follows:
   $\hat{\alpha} \pm t_{(n-p,1-\alpha)}\sqrt{var(\hat{\alpha})} = [lower(\hat{\alpha}), upper(\hat{\alpha})]$
   $\hat{\beta} \pm t_{(n-p,1-\alpha)}\sqrt{var(\hat{\beta})} = [lower(\hat{\beta}), upper(\hat{\beta})]$
   where p is the number of the parameters of ERD, n is the number of the sample size, α is the significance level, $t_{(n-p,1-\alpha)}$ is the tabular value of the t-test and var$(\hat{\alpha})$, var$(\hat{\beta})$ are obtained from the Jacobian matrix:
   $J = \begin{bmatrix} \frac{\partial f(\alpha)}{\partial \alpha} & \frac{\partial f(\alpha)}{\partial \beta} \\ \frac{\partial f(\beta)}{\partial \alpha} & \frac{\partial f(\beta)}{\partial \beta} \end{bmatrix} \equiv \begin{bmatrix} var(\hat{\alpha}) & cov(\hat{\alpha}, \hat{\beta}) \\ cov(\hat{\alpha}, \hat{\beta}) & var(\hat{\beta}) \end{bmatrix}$

4. Using the CIE to generate the fuzzy parameters depends on the pentagonal membership function as follows:
   $\tilde{\alpha} = [lower(\hat{\alpha}), \bar{X} - S^2, \bar{X}, \bar{X} + S^2, upper(\hat{\alpha})]$
   $\tilde{\beta} = [lower(\hat{\beta}), \bar{X} - S^2, \bar{X}, \bar{X} + S^2, upper(\hat{\beta})]$,
   where $\bar{X} = \frac{\sum_{i=1}^{n} x_i}{n}$ and $S^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{X})}{n}$.

5. Now, substitute the fuzzy numbers for the fuzzy parameters $(\tilde{\alpha}, \tilde{\beta})$ into the nonlinear ranking functions that we found from NL-PMF and get the crisp parameters respectively.

6. Find the survival function $S(t)$ after fuzzy.

Table 1: The MSE values for the survival function (before fuzzy) with different initial values, $n = 54$

$\alpha_0 = 0.5, \beta_0 = 0.1$	$\alpha_0 = 0.02, \beta_0 = 0.01$	$\alpha_0 = 0.08, \beta_0 = 0.04$
MSE	MSE	MSE
0.3485192	0.4108162	0.3647961

7. Using the MSE technique for the survival function prior the fuzzy $\hat{S}(t)$ and beyond the fuzzy $S(t)$ which is given via the following equation:
   $MSE[S] = \sum_{i=1}^{n} \left[S(t_i) - \hat{S}(t_i)\right]^2 /(n - p)$,
   where $S(t_i)$ was obtained from step (6) and $\hat{S}(t_i)$ was obtained from step (2). The results are illustrated in the following tables:

Table 2: The MSE values for Survival function (after fuzzy) with different initial values, $n = 54$

$m = 4$	$\alpha_0 = 0.5, \beta_0 = 0.1$		$\alpha_0 = 0.02, \beta_0 = 0.01$			$\alpha_0 = 0.08, \beta_0 = 0.04$		
r	w	MSE_NL	r	w	MSE_NL	r	w	MSE_NL
0.1	0.2	0.1506025	0.1	0.2	0.1615069	0.1	0.2	0.1511383
0.2	0.3	0.0968118	0.2	0.3	0.1023575	0.2	0.3	0.09688115
0.3	0.4	0.0613497	0.3	0.4	0.0638357	0.3	0.4	0.0610789
0.4	0.5	0.0373281	0.4	0.5	0.0382385	0.4	0.5	0.0369925
0.5	0.6	0.0211579	0.5	0.6	0.0213646	0.5	0.6	0.0208772
0.6	0.7	0.0106274	0.6	0.7	0.0105932	0.6	0.7	0.0104498
0.7	0.8	0.0042462	0.7	0.8	0.0041837	0.7	0.8	0.0041640
0.8	0.9	0.0009598	0.8	0.9	0.0009359	0.8	0.9	0.0009374

From the results, note that:

- In table 1, when ($\alpha_0 = 0.5, \beta_0 = 0.1$) the MSE is 0.3485192, but in table 2 the MSE 0.0009598 for the same initial values which is a minimum one.
- When ($\alpha_0 = 0.02, \beta_0 = 0.01$) in table 1 the MSE is 0.4108162, and the MSE is 0.0009359 is the least in table 2 for ($\alpha_0 = 0.02, \beta_0 = 0.01$).
- The MSE is 0.3647961 with ($\alpha_0 = 0.08, \beta_0 = 0.04$) in table 1. Also, with ($\alpha_0 = 0.08, \beta_0 = 0.04$) the MSE 0.0009374 in table 2 is the smallest value.

9 Conclusions

1. When we compare the results, we find that the mean squared error after fuzzy is better than the mean squared error before fuzzy.

2. After fuzzy, we made a comparison between the mean squared errors for the different initial values of all values r and w for each the estimated values. We found that $MSE = 0.0009359$ was the minimum value when $(\alpha_0 = 0.02, \beta_0 = 0.01)$ that correspond to the estimated values $(\hat{\alpha} = 0.1705944, \hat{\beta} = 0.1747365)$.

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