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Irresolvability Ideal Module

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Abstract

In this paper we focus on the relationship between the concept of $I-$irresolvable and the filter of the $I-$dense, as well as the strongly $I-$irresolvable so as to highlight their most important benefit. We also shed some light on Hewitt's definition on these concepts and the effect of codenseness. We prove that the intersection of any two $I-$dense subsets of X does not necessarily need to be an I-dense subset and that the set X is also not necessarily $I-$dense. We conclude that the

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family of all $I-$dense subsets of X will not form a filter, stating that $T \cap I=\emptyset$ represents the necessary and sufficient condition to be a filter.

1 Introduction

In 1943, Hewitt's resolvability [1] of decomposing a space into two dense sets (otherwise the space is considered irresolvable) aroused the interest of researchers ever since due to its great importance in applied fields. Back in 1933 Kuratowski [2] had expanded the concept of limit points which was said to be the local function of the set. In turn, it was considered a broader topological space than the one before. In 1990, Jankovic and Hamlett [3] developed the basic characteristics of these points and their sets. More results on the fuzzy ideal and soft ideal topological spaces are documented in [4, 1, 5, 6]. Hewitt defined codense as the ideal codense, if all non-empty open subsets of the topological space are not members of this ideal.

2 Irresolvablity in Ideal Topological Spaces

Definition 2.1. *An Ideal Topological Space (ITS) is a triplet (X,T,I) such that:*

- *Asubset K of X [7] is I-dense if $k^* = X$, whereas*

$$K^* = \{x \in X : \forall W \in T, x \in W \text{ and } W \cap K \notin I\}.[3]$$

- *X is I-resoluable, if X can be decomposed into two I-dense sets. Otherwise it is called I-irresolvable [8].*
- *X is hereditarily I-irresolvable space, if every non-empty subsets is I-irresolvable*
- *X is I-submaximal if each of its I-dense subsets is open [10].*
- *X is strongly I-irresolvable, if open subset is I-irresolvable space.*

Therefore, if I is codense, then A is I-dense if and only if A is dense. Under this condition, it is also stated that every non-empty open set is I-dense. This leads to the fact that the space is not I-irresolvable when every I-irresolvable is resolvable. The converse is not true: The *usual topology is resolvable, but for the ideal I_C it is the collection of all countable subsets of the set of real number while the set of real numbers is not I_C-resolvable.*

- *For every I-dense subset D of X and $\forall\ x \in D$, $\{x\}$ is not open.*
- *If I is codense, then every subset K of X, $\Psi k^* subseteq \Psi\big(k\big)$, where $\Psi\big(k\big) = X - \big(x - k\big)^*$ [11]*
- *The point $x \in X$ is called $*-$ frontier point of $K \subseteq X$ if it is called $*-$frontier of k^* and $(X - K)^*$. The set of all $*-$ frontier points of K is denoted by $F^*\ \big(K\big)$ [12].*
- *It is easily shown that for any I-resolvable subspace K of X the local function of it is also an I-resolvable Subspace. In addition, the union of any family of I-resolvable is I-resolvable [9].*

Proposition 2.2. *The ITS $\big(X,T,I\big)$ is I-irresolvable if and only if there is no I-dense subset D of X for which X-D is also I-dense.*

Proposition 2.3. *If the ITS $\big(X,T,I\big)$ is I-resolvable and door space (every subset is open or closed or both), then (X,T) is T_1-space.*

Proof.

By assumption, there exist disjoint subsets D_1,D_2 of X such that $X = D_1 \cap D_2$ and D_1^*=D_2^*=X.
This implies that $\forall$ x $\in X$ $x \in D_1$. So $x \notin D_2$ and hence $\{x\}$ is not open. B But X is a door space and thus $\{x\}$ is closed. Similarly, for $x \in D_2$ $\{x\}$ is closed.

For ITS (X, T, I), it is obvious that every l-dense subset H of X is l-dense if and only if int $(H^*) \subseteq [int(H)]^* \subseteq [\psi(H)]^*$, assuming that I is condense whereby int (H) is the interior of the get H.

- The important question that arises is: under what condition does the family IDX of l-dense subsets of the ITS (X, T, I) form a filter?

Proposition 2.4. *For ITS (X, T, l), every l-dense subset H of X, $\mathrm{int}(H)$ is l-dense. Then the intersection of two l-dense is l-dense.*

From this proposition, we see that the family of all I-dense is a filter generated the open subsets of X.

Also, for l is codense, the family $DID(X) = \{A \in T : A^* = X\}$ is a filter.

Proposition 2.5. *For any ITS (X,T,I) with I is codense and $ID(X)$ is a filter, $\forall \varnothing \neq W \in T, (W, T_W, I_W)$ is l-irresolvable.*

Proof.

If $\emptyset \neq W \in T$ is l-resolvable, then $\exists W_1 \cup W_2 = W, W_1 \cap W_2 = \emptyset$ and $W \subseteq W_1^*, W \subseteq W_2^* \Rightarrow X = X^* = (W \cup (X-W))^+ \subset W_1^* \cup (X-W)^*$. Thus $(W_1 \cup (X-W)$ and $W_2 \cup (X-W)$ a disjoint l-dense subset of X and $X = [(W_1 \cap W_2) \cup (X-W)]^* \Rightarrow X = (X-W)' \subset X - W \Rightarrow W = \varnothing$ which is a contradiction.

Definition 2.6. *The ITS (X,T,I) is called a T-l-resolvable if $\exists$ disjoint I-dense K and T-dense $H \ni X = K \cup H$; otherwise, it is called T-l-irresolvable.*

Proposition 2.7. *For any ITS $(X,T,l), \forall W \in T, (W, T_W, I_W)$ is $T_W - T_w$-irresolvable, if $H^* = X$, then $\mathrm{cl}(\mathrm{int}(H)) = X$.*

Proof.

Suppose that $\exists W \in T \ni W \cap \mathrm{int}(H) = \emptyset$. Since $W = (W \cap H) \cup (W - H)$ and $W = W \cap H^* \subset (W \cap H)^*, W \subseteq \mathrm{cl}(W - H) \Rightarrow W \cap H$ and $W - H$ are disjoint non-empty I-dense and T-dense respectively of W, this conducts that W is $T_W - I_{W^-}$-resolvable.

Proposition 2.8. *Let ITS (X,T,I) be the decomposition by T-closed Y_2 and any subset Y_1, of X. If (Y_1, T_{Y_1}, I_{Y_1}) and (Y_2, T_{Y_2}, I_{Y_2}) re hereditarily I-irresolvable, then (X,T,I) is hereditarily I-irresolvable.*

Proof.

For $\emptyset \neq M \subseteq X, (M, T_M, I_M)$ is I-resolvable.
$\Rightarrow \exists$ two disjoint I_M-dense M_1, M_2 of $M \ni M = M_1 \cup M_2$.
Since Y_1 is closed, Y_2 is open.
If possible $Y_1 \cap M_1 \neq \emptyset$ and $Y_2 \cap M_2 \neq \emptyset$
$\Rightarrow \forall V \in T \ni V \cap M \cap M_1 \notin I_M$, but $Y_2 \cap V \in T$
$\Rightarrow Y_2 \cap V \cap M \cap M_1 \notin I_M$
$\Rightarrow V \cap (Y_2 \cap M) \cap (Y_2 \cap M_1) \notin I_{M \cap Y_2}$.
Then $(Y_2 \cap M_1)^*_{M \cap Y_2} = M \cap Y_2$, Similarly $(Y_2 \cap M_2)^*_{M \cap Y_2} = M \cap Y_2$.
So $(Y_2 \cap M_1)$ and $(Y_2 \cap M_2)$ are the disjoint union of I-dense of $M \cap Y_2$, which contradicts the assumption. Therefore, either $M_1 \cap Y_2 = \emptyset$ or $M_2 \cap Y_2 = \emptyset$.
For $M_1 \cap Y_2 = \emptyset \Rightarrow M_1 \subset X$ and $M_2 \cap Y_2 \neq \emptyset$.
But $M^*_{1_M} = M \Rightarrow \forall W \in T \ni W \cap M \cap M_1 \notin I_M$
$\Rightarrow W \cap M \cap Y_1 \notin I_M \Rightarrow (M \cap Y_1)^*_M = M$.
And $M = (M \cap Y_1)^* \cap M \subseteq M^* \cap Y_1^* \cap M \subseteq Y_1$. This means that M is the I-resolvable subspace of X in this contradiction. The same is true for $M_2 \cap Y_2 = \emptyset$ and $M_1 \cap Y_2 \neq \emptyset$. This final contradiction proves that (X,T,I)

is hereditarily I-irresolvable.
For any ITS (X, T, I) with I is codense, every I-submaximal is hereditary I irresolvability since taking any non-empty subset M of X which is I resolvable and using codense will eventually lead to a contradiction.

Theorem 2.9. *Let (X, T, I) be strongly I-irresolvable with l being codense. The following statements are equivalent:*

1. *Each open subspace is I-irresolvable.*
2. *$\forall I$-dense M of X, $[\psi(M)]^* = X$.*
3. *$\forall I$-codense M of X, $\psi\left(M^+\right) =$*
4. *Every clopen subset $M = X \ni M = H \cup K, H \in T - \psi\left(K^*\right) = \emptyset$*

Proof.
$1) \Rightarrow 2)$.
Let $M^* = X$. $W = X - [\psi(M)]^* \neq \emptyset \Rightarrow W = W \cap M^* \subseteq (W \cap M)^*$. But $\psi(M) \subseteq [\psi(M)]'$ by the property of codenseness. Then $X/[\psi(M)]^* \subseteq [X/M]^*$.
Thus $W \subset [X/M]^* \Rightarrow W = W \cap [X/M]^* \subseteq [W \cap X/M]^*$.
Then W is l-resolvable which is a contradiction.

$2) \Rightarrow 3)$.
Let M is I-codense. Then $[X/M]^* = X$.
By $ii'[\psi(X/M)]^* = X$ if and only if $\psi\left(M^*\right) = \emptyset$.
$3) \Rightarrow 4)$.
For any subset M of X, either $\psi(M) = \emptyset$ or $\psi(M) \neq \emptyset$.
For $\psi(M) = \emptyset \Rightarrow [X/M]^* = X \Rightarrow X/M$ is l-codense and by iii, $\psi\left(M^*\right) = \emptyset \Rightarrow M = \cup M$.
For $\psi(M) \neq \varnothing \Rightarrow \psi[M \cap X/\psi(M)] = \varnothing$.
Otherwise, $\exists x \in \psi[M \cap X/\psi(M)] \Rightarrow \exists W \in T(*) \exists W \cap X/[M \cap X/\psi(M)] \in l \Rightarrow W \cap \psi(M) \in I$. This contradicts with $T \cap l = \varnothing$.

$\Rightarrow X/[M \cap X/\psi(M)]$ is I-dense.
Then $\psi\left[[M \cap X/\psi(M)]'\right] = \emptyset$ and $M = \psi(M) \cup [M \cap X/\psi(M)]$.
$4) \Rightarrow 1)$.
If $\exists \varnothing \neq W \in T$ is I-resolvable $\Rightarrow \exists$ two disjoint non-empty I-dense W_1, and $W_2 \exists W = W_1 \cup W_2$.
So by iv, $W_1 = V \cup N$, where $V \in T$ and $\psi\left(N^*\right) = \emptyset \Rightarrow V \neq \varnothing$ and $W \subseteq W_1^*$.

$\Rightarrow W \subseteq \psi(W) \subseteq \psi(W_1^*) = \emptyset \Rightarrow W = \emptyset$ *which is a contradiction. Consequently,* $\emptyset \neq V \neq int(V) \subseteq int_W(W_1)$
$\Rightarrow \text{int}_W(W_1) \neq \emptyset$*, but* W_2 *is* I*-dense in* W
$\Rightarrow W_1 \cap W_2 \neq \varnothing$ *which is a contradiction.*

3 Conclusion

We observed that the intersection of any two I-dense subsets of X does not necessarily require the subset and the set X to be I-dense. Therefore, the family of all I-dense subsets of X will not be a filter, as the necessary and sufficient condition for it to be a filter is $T \cap I = \emptyset$. It is recommended to study the concepts that are the basis of this study in the i-topological spaces on C-topological space.

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