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Cyclic Intersection Graph of Subgroups of Dihedral Groups and its Properties

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Abstract

Various characteristics of the algebraic structures can be expressed by associating it to the graphs of different groups. In an intersection graph, each vertex conforms to a set wherein two vertices are connected by an edge if and only if their corresponding sets have a non-empty intersection. For a finite group G, a graph of its subgroups

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can be represented by the vertices that correspond to the subgroups of G. Based on this factor, we defined the cyclic intersection graph of the subgroups of a group G. The cyclic intersection graph of the subgroups of a group G enclosed all the nontrivial subgroups as its vertices together with two adjacent vertices H and K if and only if H intersection K corresponded to a non-trivial cyclic subgroup. Furthermore, the general presentation of the graph on the dihedral groups was obtained. The cyclic intersection graph of the subgroups on dihedral groups is classified as connected or planar. Additionally, some of the invariants of the the cyclic intersection graph of the subgroups on the dihedral groups are given.

1 Introduction

In recent years, numerous approaches have been adopted to study various characteristics of a group wherein the graphs associated to diverse algebraic structures were the main focus. In this regard, the concepts of a group and its geometric properties for defining the graphs were used as one of the most versatile techniques. This conceptualization became significant to bridge the gap between the graph and group theory, thus achieving the attributes of the graphs related to different groups. The notion of groups involving the graph theory was first introduced by Cayley [1] wherein the graph that interprets the abstract structure of a group generated by a set of generators was defined. Diverse graphs related to the algebraic structures can be used to determine their special features. In this perception, the intersection graphs have extensively been explored over the last few decades. For instance, Bosak [2] examined the intersection graphs of the semigroups. Csakany and Pollak [3] studied the intersection graphs of the subgroups of a finite group. Shen [4] classified the finite groups with disconnected intersection graphs of the subgroups. In addition, Zelinka [5] obtained the independence number of the intersection graph of subgroups of a finite abelian group in which it was acknowledged that two finite abelian groups with isomorphic intersection graphs can be isomorphic. Uehara [6] investigated the geometrical intersection graphs. Later, Kayacan and Yaraneri [7] verified the claim made by Zelinka [5]. intersection graphs for every group were determined. Meanwhile, the lower bounds of the isoperimetric numbers of the random intersection graphs induced by these perturbations were evaluated by Shang [8]. Briefly, the intersection graphs on diverse algebraic structures including the rings, vector spaces, and modules were examined in-depth. Based on these revelations, we proposed a new type of cyclic intersection graph of the subgroups of finite groups for the first time. The general presentations for the graph on the dihedral groups were portrayed.

2 Preliminaries

This section describes some basic concepts, notations, and preliminaries related to this work. We used a type of finite group called dihedral group.

Definition 2.1. *[9] Dihedral Group*
For $n \in \mathbb{Z}$ and $n \geq 3$, the dihedral group, D_{2n}, is the set of symmetries of a regular n-gon. Furthermore, the order of D_{2n} is $2n$ or equivalently $|D_{2n}| = 2n$. The dihedral group can be defined in terms of generators and relations as follows:

$$D_{2n} = \langle a, b \mid a^n = b^2 = e, bab = a^{-1} \rangle.$$

Herein, simple undirected graphs without loop or multiple edges are considered. The sets of vertices and edges of a graph Γ can be expressed as $V(\Gamma)$ and $E(\Gamma)$, respectively. The adjacency of vertices a, b is written as $a \sim b$, the number of vertices of the graph Γ is denoted as $|V(\Gamma)|$, the degree of the vertex v is presented as $deg(v)$, and the largest vertex degree (the maximum degrees of the vertices of a graph) is expressed as $\Delta(\Gamma)$. A graph Γ is said to be connected if there is a path between every pair of its vertices. Meanwhile, a graph is said to be complete if there is an edge between every pair of its vertices. A graph is considered to be planar if it can be drawn in a plane without edge crossing. A clique is defined as a subset U of the vertices of Γ such that the induced subgraph of U corresponds to a complete graph. The maximum size of a clique is referred to as the clique number of Γ that is denoted as $\omega(\Gamma)$. The girth of a graph Γ represents the length of the shortest cycle enclosed in the graph. If Γ has no cycle then the girth of the graoh is infinity. The diameter of a graph represents the maximum distance between the pair of its vertices. It is ∞ if the graph is disconnected. The vertex chromatic number of a graph Γ denoted as $\chi(\Gamma)$ characterizes the smallest number of colors needed to color the vertices of Γ such that no two adjacent vertices get the same color. The number of colors required to color the edges of the graph in such a way that no two adjacent edges have the same color is the edge chromatic number of a graph represented as $\chi^e(\Gamma)$. The upcoming section explains some of these graphs belong to the finite groups.

Definition 2.2. *[10] Intersection Graph of a Group*
Let, $\mathcal{F} = \{s_i \mid i \in I\}$ *be an arbitrary family of sets. The intersection graph of* $\mathcal{F}$ *is a graph having the elements of* $\mathcal{F}$ *as its vertices and two vertices* s_i *and* s_j *are adjacent if and only if* $i \neq j$ *and* $s_i \cap s_j \neq \emptyset$*.*

Definition 2.3. *[3] The Intersection Graph of Subgroups of a Group*
Let G *be a group. The intersection graph of subgroups of* G*, denoted by* $\mathcal{J}(G)$ *is a graph with all proper subgroups of* G *as its vertices and two distinct vertices in* $\mathcal{J}(G)$ *are adjacent if and only if the corresponding subgroups have a non-trivial intersection in* G*.*

3 Results

In this study, firstly the cyclic intersection graph of subgroups of a group is introduced and the graph for the dihedral groups is constructed. Next, the general presentation of the graphs is examined using the earlier definitions and results. The formal definition of the cyclic intersection graph of subgroups of a group is given as in Definition 3.1.

Definition 3.1. *The Cyclic Intersection Graph of Subgroups of Finite Groups*
Let G *be a group. The cyclic intersection graph of subgroups of* G *denoted by* $\Gamma^c_\cap(G)$ *represents a graph having all non-trivial subgroups as its vertices and two vertices* H *and* K *are adjacent iff* H *intersection* K *is a non-trivial cyclic subgroup.*

3.1 General Presentations of Cyclic Intersection Graph of Subgroups of Dihedral Groups

The general presentations of the cyclic intersection graph of subgroups of dihedral groups are given as follows.

Theorem 3.2. Let G be a group, then the improper subgroup of G in $\Gamma^c_\cap(G)$ is adjacent to all cyclic subgroups of G.

Proof. Since G itself is the improper subgroup of G then the intersection of the improper subgroup G with all the cyclic subgroups of G becomes equal to those cyclic subgroups. Thus, there are edges linking the vertex G that is an improper subgroup with all the cyclic subgroups of the group G. □

The general presentation of the cyclic intersection graph of subgroups of D_{2p} for prime p is given in Theorem 3.3.

Theorem 3.3. *Let D_{2p} be the dihedral group of order $2p$ where p is a prime number. Then $\Gamma^c_\cap(D_{2p}) = K_{1,p+1}$.*

Proof. Let D_{2p} be a dihedral group where p is a prime number, then $D_{2p} = \langle a, b | a^p = b^2 = e, bab = a^{-1}\rangle = \{e, a, a^2, a^3, ..., a^{p-1}, b, ab, a^2b, ..., a^{p-1}b\}$. Therefore, $O(a^i) = \dfrac{O(a)}{gcd(i, O(a))} = \dfrac{p}{gcd(i, p)} = p$, and $O(a^ib) = 2$ for all $i = 0, 1, 2, ..., p-1$. Meanwhile, all the subgroups of D_{2p} take the form: $\langle e\rangle = \{e\}, H_i = \langle a^ib\rangle = \{e, a^ib\} \cong \mathbb{Z}_2$, and $H_p = \langle a^i\rangle = \{e, a, a^2, ..., a^{p-1}\} \cong \mathbb{Z}_p$ for all $i = 0, 1, 2, ..., p-1$. Explicitly, the subgroups of D_{2p} are $\{\langle e\rangle, H_0, H_1, ..., H_{p-1}, H_p, D_{2n}\}$. Therefore, the number of subgroups of D_{2p} is $p+3$
Let the vertex set, $V(\Gamma(G)) = \{H_0, H_1, ..., H_{p-1}, H_p, D_{2n}\}$. Then
Fact 1: D_{2p} is adjacent to all the subgroups $H_0, H_1, ..., H_{p-1}, H_p$, since $D_{2n} \cap H_i = H_i \cong \mathbb{Z}_2$ for $i = 0, 1, 2, ..., p-1$ and $D_{2n} \cap H_p = H_p \cong \mathbb{Z}_p$. Here, all H_i, $0 \le i \le p$ are the cyclic subgroups.
Fact 2: $H_i \cap H_j = \{e, a^ib\} \cap \{e, a^jb\} = \{e\}$. Also, $H_i \cap H_p = \{e, a^ib\} \cap \{e, a, a^2, a^3, ..., a^{p-1}\} = \{e\}$. Hence, the edges between H_i and H_j as well as between H_i and H_p for all $0 \le i \le p-1$ and $0 \le j \le p-1$ are completely absent, indicating that $\Gamma^c_\cap(D_{2p})$ is $K_{1,n+1}$. □

The general presentations of the cyclic intersection graph of subgroups of dihedral groups, for $n = p^r$ is given in the Theorem 3.4.

Theorem 3.4. *Let D_{2n} be the dihedral group of order $2n$ with $n = p^r$ where p is a prime number, $r \in \mathbb{N}$, and $r > 1$. Then $\Gamma^C_\cap(D_{2p^r})$ has subgraphs isomorphic to $K_{r+\ p^{r-1}}$, $K_1 + \overline{K}_{p^r}$ and $K_r \cup \overline{K}_{p^r}$.*

Proof. Let D_{2n} be the dihedral group of order $2n$, for $n = p^r$ where p is a prime number, $r \in \mathbb{N}$, and $r > 1$. Let $A = \langle a^t\rangle$ be the set of subgroups generated by the rotation elements of D_{2n} where $t \mid n$, $t = 1, p, p^2, p^3, ..., p^{r-1}$. In addition, let $B = \langle a^ib\rangle$ with $0 \le i \le n-1$ be the set of the subgroups generated by the reflection elements of D_{2n}. Let $C = \langle a^{p^j}, a^ib\rangle$ where $0 \le i \le (p^j)$ and $1 \le j \le r-1$.
To prove that $\Gamma\ ^C_\cap(D_{2p^rq})$ has a subgraph isomorphic to $K_{r+p^{r-1}}$, assume the subgroups $A = \langle a^t\rangle$ of D_{2n} be a clique of size r. Furthermore, consider $C_1 = \langle a^{p^{r-1}},\ a^ib\rangle \in C$, $0 \le i < p^{r-1}$ and pick arbitrary vertices $\langle a^{p^{r-1}},\ a^kb\rangle, \langle a^{p^{r-1}},\ a^mb\rangle \in C_1$ for $0 \le k < p^{r-1}$and $0 \le m < p^{r-1}$. Then $\langle a^{p^{r-1}},\ a^kb\rangle \cap \langle a^{p^{r-1}},\ a^mb\rangle = \langle a^{p^{r-1}}\rangle$ is a cyclic subgroup of order p^{r-1}. Conversely, $C_1 \cap \langle a^t\rangle = \langle a^{p^{r-1}}\rangle$ is the cyclic subgroup. Therefore, $\Gamma^C_\cap(D_{2p^r})$ contains the subgraph $K_{r+p^{r-1}}$. Next, the order of each B takes the form $|\langle a^ib\rangle| = 2$ for $0 \le i \le p^{r-1}$. Thus, $D_{2p^r} \cap \langle a^ib\rangle = \langle a^ib\rangle$ indicates that

$|D_{2p^r} \cap \langle a^i b \rangle| = 2 \cong \mathbb{Z}_2$, displaying that the intersection is a cyclic subgroup. In short, Theorem 3.2 implies that there exists an edge joining D_{2p^r} to each $\langle a^i b \rangle$. Thus, the intersection of each subgroup in B is a trivial subgroup, affirming that $\langle a^i b \rangle$ form an independent set of order p^r. In essence, the graph contains $K_1 + \overline{K}_{p^r}$. Assume that the subgroup A is a clique of size r and the subgroup B form an independent set of order p^r. Thus, the graph contains $K_r \cup \overline{K}_{p^r}$ since there are no edges between the set A and B. $\square$

Before providing the general presentations of the cyclic intersection graph of subgroups of the dihedral groups, D_{2n} for $n = pq$, Example 3.5 is given first.

Example 3.5. *Consider the dihedral group of order 12,* $D_{12} = \{e, a, a^2, a^3, a^4, a^5, b, ab, a^2b, a^3b, a^4b, a^5b\}$. *The subgroups of* D_{12} *are* $\langle e \rangle$, $\langle a \rangle$, $\langle a^2 \rangle$, $\langle a^3 \rangle$, $\langle a^2, b \rangle$, $\langle a^2, ab \rangle$, $\langle a^3, b \rangle$, $\langle a^3, ab \rangle$, $\langle a^3, a^2b \rangle$, $\langle b \rangle$, $\langle ab \rangle$, $\langle a^2b \rangle$, $\langle a^3b \rangle$, $\langle a^4b \rangle$, $\langle a^5b \rangle$ *and* D_{12}. *The vertex set is given as* $V(\Gamma(D_{12})) = \{\langle a \rangle, \langle a^2 \rangle, \langle a^2, b \rangle, \langle a^2, ab \rangle, \langle a^3, b \rangle, \langle a^3, ab \rangle, \langle a^3, a^2b \rangle, \langle b \rangle, \langle ab \rangle, \langle a^2b \rangle, \langle a^3b \rangle, \langle a^4b \rangle, \langle a^5b \rangle, D_{12}\}$. *The cyclic intersection graph of subgroups of* D_{12} *has subgraphs isomorphic to* $K_1 + (K_2 \cup \overline{K}_6)$ *and* K_6 *as shown in Figure 1 . Figure 2 illustrates the structure of* $\Gamma^C_\cap(D_{12})$.

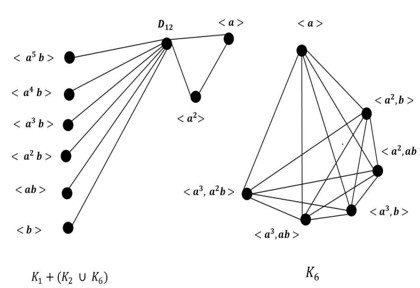

Figure 1: Subgraphs of the cyclic intersection graph of subgroups of D_{12}

Theorem 3.6. *Let* D_{2n} *be the dihedral group of order* $2n$, *for* $n = pq$ *where* p *and* q *are distinct prime numbers. Then* $\Gamma^C_\cap(D_{2pq})$ *has the subgraphs isomorphic to* $K_1 + (K_2 \cup \overline{K}_n)$ *and* K_{p+q+1}.

Proof. Suppose D_{2n} is the dihedral group of order $2n$ for $n = pq$. Let $A = \langle a^t \rangle$ be the set of subgroups generated by the rotation elements of D_{2n} where $t < n$ and $t \mid n$. In addition, let $B = \langle a^i b \rangle$ where $0 \le i \le n-1$ be

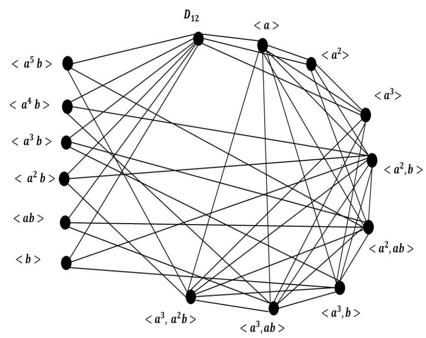

Figure 2: The cyclic intersection graph of subgroups of D_{12}

the set of the subgroups generated by the reflection elements of D_{2n}. Assume that $C = \langle a^t, a^i b\rangle$ where $0 \le i \le t$ and $1 < t < n$. To prove that $\Gamma_\cap^C(D_{2pq})$ has a subgraph isomorphic to $K_1 + (K_2 \cup \overline{K}_n)$. Consider a subset A_α of A such that $A_\alpha = \langle a^\alpha\rangle$ where $\alpha \mid p$. Then A_α is a clique of size 2. Because $A_\alpha \cap D_{2n} = A_\alpha$ is a cyclic subgroup and thus $\langle a^i b\rangle \cap D_{2n} = \langle a^i b\rangle$ is also a cyclic subgroup, indicating that the edges linking D_{2n} with both A_α and $\langle a^i b\rangle$ exist. Therefore, $\Gamma_\cap^C(D_{2pq})$ has a subgraph isomorphic to $K_1 + (K_2 \cup \overline{K}_n)$. To further verify that the graph has a subgraph isomorphic to K_{p+q+1}, let us consider the non-cyclic subgroups $C_1 = \langle a^p, a^i b\rangle$, $C_2 = \langle a^q, a^j b\rangle$ of C and $A_1 = \langle a\rangle$ of A (for $0 \le i \le (p-1)$, $0 \le j \le q-1$). Then both $C_1 \cap C_2 \in B$ and $C_1 \cap A_1 = \langle a^p\rangle$ are the cyclic subgroup, demonstrating that $C_2 \cap A_1 = \langle a^q\rangle$ is also a cyclic subgroup. Thus, C_1, C_2 and A_1 with $\{\langle a^p, a^i b\rangle, \langle a^q, a^j b\rangle, \langle a\rangle\}$ is a clique of size $p + q + 1$. This reaffirms that $\Gamma_\cap^C(D_{2p^rq})$ contains the subgraph K_{p+q+1}. □

The general presentations of the cyclic intersection graph of subgroups of dihedral groups, for $n = p^r q$ is given in the Theorem 3.7.

Theorem 3.7. *Let D_{2n} be the dihedral group of order $2n$, for $n = p^r q$ where p and q are distinct prime numbers and $r > 1$ for $r \in \mathbb{N}$. Then $\Gamma_\cap^C(D_{2p^rq})$ encloses the subgraphs isomorphic to $K_1 + (K_\alpha \cup \overline{K}_n)$ where α are the positive divisors of p^r and*

$$\begin{cases} K_{\beta+\gamma} & if\ p = 2, \\ K_{p^r+q+r} & if\ p \neq 2, \end{cases}$$

where β is the greatest proper divisor of n and γ is the positive divisor of β.

Proof. Suppose D_{2n} is the dihedral group of order $2n$ for $n = p^r q$. Let $A = \langle a^t \rangle$ be the set of subgroups generated by the rotation elements of D_{2n} where $t < n$, and $t \mid n$. In addition, let $B = \langle a^i b \rangle$, $0 \leq i \leq n-1$ be the set of subgroups generated by the reflection elements of D_{2n}. To prove that $\Gamma^C_\cap(D_{2p^rq})$ where $r > 1$ for $r \in \mathbb{N}$ has a subgraph isomorphic to $K_1 + (K_\alpha \cup \overline{K}_n)$ consider a subset A_α of A such that $A_\alpha = \langle a^\alpha \rangle$ where $\alpha \mid p^r$. Then A_α is a clique of size α. Now, because $A_\alpha \cap D_{2n} = A_\alpha$ is a cyclic subgroup, then $\langle a^i b \rangle \cap D_{2n} = \langle a^i b \rangle$ is also a cyclic subgroup. Therefore, the edges linking D_{2n} with both A_α and $\langle a^i b \rangle$ exist. Accordingly, $\Gamma^C_\cap(D_{2p^rq})$ contains a subgraph isomorphic to $K_1 + (K_\alpha \cup \overline{K}_n)$. To show that the graph has a subgraph isomorphic to $K_{\beta+\gamma}$ if $p = 2$, consider the non-cyclic subgroups $C = \langle a^\beta, a^i b \rangle$ where β is the greatest proper divisor of n and $0 \leq i \leq (\beta - 1)$ and select some arbitrary $C_1 = \langle a^\beta, a^{i_1} b \rangle$ and $C_2 = \langle a^\beta, a^{i_2 b} \rangle$ in $C = \langle a^\beta, a^i b \rangle$. Then $C_1 \cap C_2 = \langle a^\beta \rangle$ is a cyclic subgroup of order $|\beta|$. Let $A_1 = \langle a^\gamma \rangle \in A$. Because γ is the positive divisor of β, therefore, $\langle a^\gamma \rangle \cap C = \langle a^\beta \rangle$, indicating that the edges linking C with A_1 exist. Hence, $\Gamma^C_\cap(D_{2p^rq})$ contains the subgraph $K_{\beta+\gamma}$ when $p = 2$. If $p \neq 2$ and the graph encloses a subgraph isomorphic to K_{p^r+q+r}, then consider the non-cyclic subgroups $C_1 = \langle a^{p^r}, a^i b \rangle$, $C_2 = \langle a^q, a^j b \rangle$ and $A_1 = \langle a^y \rangle$ where y is proper divisors of p^r for $0 \leq i \leq (p^r - 1)$, $0 \leq j \leq q - 1$ and $1 \leq y \leq r$. Then, $C_1 \cap C_2 \in B$, $C_1 \cap A_1 = \langle a^{p^r} \rangle$ and $C_2 \cap A_1 = \langle a^{qy} \rangle$ are the cyclic subgroup. Consequently, C_1, C_2 and A_1 with $\{\langle a^{p^r}, a^i b \rangle, \langle a^q, a^j b \rangle, \langle a^y \rangle\}$ is a clique of size $p^r + q + r$. Briefly, $\Gamma^C_\cap(D_{2p^rq})$ contains the subgraph K_{p^r+q+r}. □

The general presentations of the cyclic intersection graph of subgroups of dihedral groups, for $n = pqh$ is given in the Theorem 3.8.

Theorem 3.8. *Let D_{2n} be the dihedral group of order $2n$, for $n = pqh$ where p, q and h are distinct prime numbers and p is the smallest prime number between them. Then $\Gamma^C_\cap(D_{2pqh})$ contains the subgraphs isomorphic to $K_1 + (K_4 \cup \overline{K}_n)$ and $K_{\frac{n}{p}+4}$.*

Proof. Let $\langle a^i \rangle$ and $\langle a^i b \rangle$ be the subgroups generated by the rotation elements and reflection elements of D_{2n} where $n = pqh$. According to Lagrange's Theorem p, q and h divide $|D_{2n}|$. Thus, the subgroups $\langle a^p \rangle$, $\langle a^q \rangle$ and $\langle a^h \rangle$ in D_{2n}, such that $\langle a^p \rangle \cap \langle a^q \rangle = \langle a^{pq} \rangle$, $\langle a^p \rangle \cap \langle a^h \rangle = \langle a^{ph} \rangle$, $\langle a^q \rangle \cap \langle a^h \rangle = \langle a^{qh} \rangle$, $\langle a^p \rangle \cap \langle a \rangle = \langle a^p \rangle$, $\langle a^q \rangle \cap \langle a \rangle = \langle a^q \rangle$, and $\langle a^h \rangle \cap \langle a \rangle = \langle a^h \rangle$ exist. Hence, $\{\langle a \rangle, \langle a^p \rangle, \langle a^q \rangle, \langle a^h \rangle\}$ is a clique of size 4. Moreover, $\langle a^p \rangle \cap \langle D^{2n} \rangle = \langle a^p \rangle$, $\langle a^q \rangle \cap \langle D^{2n} \rangle = \langle a^q \rangle$, $\langle a^h \rangle \cap \langle D^{2n} \rangle = \langle a^h \rangle$ and $\langle a \rangle \cap \langle D^{2n} \rangle = \langle a \rangle$. Thus, the edges linking D_{2n} with $\{\langle a \rangle, \langle a^p \rangle, \langle a^q \rangle, \langle a^h \rangle\}$ exist thereby forming $K_1 + K_4$.

Now, because $\langle a^i b\rangle$ is an independent set, thus it forms a cyclic subgroup $\overline{K_n}$. However, $\langle a^i b\rangle \cap \langle D^{2n}\rangle = \langle a^i b\rangle$ being a cyclic subgroup it can form $K_1 + \overline{K_n}$. By combining $K_1 + K_4$ and $K_1 + \overline{K_n}$ one gets the subgraph $K_1 + (K_4 \cup \overline{K_n})$. In order to show that $K_{\frac{n}{p}+3}$ is also a subgraph of $\Gamma^C_\cap(D_{2pqh})$ consider the non-cyclic subgroup $\langle a^{qh}, a^i b\rangle$ for $0 \le i \le (qh-1)$ generated by the joint elements of the rotations and reflections of D_{2n}. Then $\langle a^{qh}, a^i b\rangle \cap \langle a\rangle = \langle a^{qh}\rangle$, $\langle a^{qh}, a^i b\rangle \cap \langle a^q\rangle = \langle a^{qh}\rangle$, $\langle a^{qh}, a^i b\rangle \cap \langle a^h\rangle = \langle a^{qh}\rangle$ and $\langle a^{qh}, a^i b\rangle \cap \langle a^{qh}\rangle = \langle a^{qh}\rangle$. As a result, $\{\langle a^{qh}, a^i b\rangle, \langle a\rangle, \langle a^q\rangle, \langle a^h\rangle, \langle a^{qh}\rangle\}$ is a clique of size $qh+4$. Hence, the graph contains $K_{\frac{n}{p}+4}$ as a subgraph. □

3.2 Classification of Cyclic Intersection Graph of Subgroups of Dihedral Groups

The classifications of the cyclic intersection graph of subgroups of dihedral groups are given in Proposition 3.9 and Proposition 3.10.

Proposition 3.9. *Let D_{2n} be the dihedral group of order $2n$. Then the cyclic intersection graph of subgroups of dihedral groups is connected.*

Proof. According to Theorem 3.2, the subgroup D_{2n} is adjacent with all cyclic subgroups in D_{2n}. Now, consider the non-cyclic subgroup $\langle a^t,\ a^i b\rangle$, where $t \mid n$, and $0 \le i \le t$ is connected. Pick arbitrary non-trivial elements $\langle a^{t_1},\ a^{i_1} b\rangle, \langle a^{t_1}, a^{i_2} b\rangle \in \langle a^t, a^i b\rangle$, then $\langle a^{t_1}, a^{i_1} b\rangle \cap \langle a^{t_1}, a^{i_2} b\rangle = \langle a^{t_1}\rangle$. Furthermore, $\langle a^{t_1}, a^{i_1} b\rangle \sim \langle a^{t_1}, a^{i_2} b\rangle$. Since $\langle a^t\rangle \cap \langle a^t,\ a^i b\rangle = \langle a^t\rangle$ then $\langle a^t\rangle \sim \langle a^t,\ a^i b\rangle$. □

Proposition 3.10. *Let D_{2n} be the dihedral group where $n = p^r$ then $\Gamma^C_\cap(D_{2p^r})$ is planar if $r = 1$ for any prime p and if $r = 2$ for $p = 2$.*

Proof. Suppose D_{2n} is the dihedral group of order $2n$, for $n = p^r$ and $r = 1$. Then according to Theorem 3.3, $\Gamma^C_\cap(D_{2p^r})$ is the star graph. Thus, it does not have a subdivision of K_5 or $K_{3,3}$ that is planar. According to Theorem 3.4, $\Gamma^C_\cap(D_{2p^r})$ has a subgraph isomorphic to $K_{r+p^{r-1}}$ that is the maximal subgraph of the graph. Now, if $r = 2$, and $p = 2$, then the maximal subgraph K_4 is still planar. □

3.3 Properties of Cyclic Intersection Graph of Subgroups of Dihedral Groups

This subsection presents the results on the maximum degrees, clique number, girth, diameter, chromatic numbers, and the edge chromatic numbers of the cyclic intersection graph of subgroups on the dihedral groups.

In Proposition 3.11, the maximum degrees of the vertices of the cyclic intersection graph of subgroups on dihedral groups are found.

Proposition 3.11. Let D_{2n} be the dihedral group of order $2n$. Then the maximum degrees of the vertices of the cyclic intersection graph of subgroups of D_{2n} where $n = p, p^r, p^r q$ and pqh can be expressed as:

$$\Delta(\Gamma_{\cap}^{C}(D_{2n})) = \begin{cases} n+d & if \quad n = p^r \quad or \quad n = pq, \\ (\Sigma t)+d & if \quad n = p^r q \quad or \quad n = pqh, \quad and \quad r > 1 \end{cases}$$

where p, q and h are the distinct prime numbers, $r \in \mathbb{N}$, d is the number of proper positive divisors of n and Σt is the sum of the non-trivial proper divisors of n.

Proof. Suppose D_{2n} is the dihedral group of order $2n$. Then D_{2n} has d subgroups of type $\langle a^t \rangle$ where $t \neq n$ and $t \mid n$. Also, D_{2n} has n subgroups of type $\langle a^i b \rangle$ for $0 \leq i \leq n$ and Σt subgroups of type $\langle a^t, a^j b \rangle$ for $0 \leq j \leq t-1$. Now if $n = p^r$ or $n = pq$, then by Theorem 3.2, D_{2n} is adjacent with all the cyclic subgroups, indicating the existence of edges linking D_{2n} with $\langle a^t \rangle$ and $\langle a^i b \rangle$. However, $\langle a^t, a^j b \rangle \cap D_{2n} = \langle a^t, a^j b \rangle$ being a non-cyclic subgroup the edges between $\langle a^t, a^j b \rangle$ and D_{2n} are absent. Thus, most of the vertices of the graph are reachable to one another through D_{2n}. Consequently, the maximum degree is $|\langle a^t \rangle| + |\langle a^i b \rangle| = d+n$. If $n = p^r q$ or $n = pqh$, $r > 1$. Then $\langle a \rangle \sim D_{2n}$, $\langle a \rangle \sim \langle a^t \rangle$ for $1 < t < n$, and $\langle a \rangle \sim \langle a^t, a^j b \rangle$. Thus, the vertices D_{2n}, $\langle a^t \rangle$ and $\langle a^t, a^j b \rangle$ are reachable to one another through $\langle a \rangle$. This demonstrates that the maximum degree occurs at the vertex $\langle a \rangle$ and hence $\langle a \rangle$ excludes from $|\langle a^t \rangle|$. As a result, $\Delta(\Gamma_{\cap}^{C}(D_{2n})) = |\langle a^t \rangle| + |\langle a^t, a^j b \rangle| + D_{2n} = (d-1) + \Sigma t + 1 = d + \Sigma t$. □

Now it is customary to present the clique number of the cyclic intersection graph of subgroups on dihedral groups using the Proposition 3.12 and 3.13.

Proposition 3.12. Let D_{2n} be the dihedral group of order $2n$ for $n = p^r$ where p is a prime number and $r > 1$ for $r \in \mathbb{N}$. Then the clique numbers of the cyclic intersection graph of subgroups on the dihedral groups can be written as:

$$\omega(\Gamma_{\cap}^{C}(D_{2n})) = \begin{cases} 2 & if \ n = p, \\ r + p^{r-1} & if \ n = p^r. \end{cases}$$

Proof. If $n = p$, then by Theorem 3.3, $\Gamma_{\cap}^{C}(D_{2p})$ is the star graph. Thus, the edges between D_{2p} and every cyclic subgroup are present. Hence $\omega(\Gamma_{\cap}^{C}(D_{2n})) = 2$. If $n = p^r$, then by Theorem 3.4, the size of the maximum clique is $r + p^{r-1}$. Therefore, $\omega(\Gamma_{\cap}^{C}(D_{2n})) = r + p^{r-1}$. □

Proposition 3.13. Let D_{2n} be the dihedral group of order $2n$. Then the clique numbers of the cyclic intersection graph of subgroups on the dihedral groups takes the form:

$$\omega(\Gamma_{\cap}^{C}(D_{2n})) = \begin{cases} p+q+1 & if \quad n = pq, \\ \beta+\gamma & if \quad n = 2^r q \qquad r > 1 \ for \ r \in \mathbb{N} \\ p+q+r & if \quad n = p^r q, \qquad p \neq 2 \\ \frac{n}{p}+4 & if \quad n = pqh, \end{cases}$$

where p, q and h are the prime numbers, β is the greatest proper divisor of n and γ is the positive divisors of β.

Proof. If $n = pq$, then by Theorem 3.6, the size of the maximum clique is $p+q+1$. Therefore, $\omega(\Gamma_{\cap}^{C}(D_{2n})) = p+q+1$. If $n = 2^r q$, then by Theorem 3.7, the size of the maximum clique is $\beta+\gamma$. Consequently, $\omega(\Gamma_{\cap}^{C}(D_{2n})) = \beta+\gamma$. If $n = p^r q$ and $n \neq 2$, then by Theorem 3.7, the size of the maximum clique is $p+q+r$. Thus, $\omega(\Gamma_{\cap}^{C}(D_{2n})) = p+q+r$. If $n = pqh$, then by Theorem 3.8, the size of the maximum clique is $\frac{n}{2}+4$. Thus, $\omega(\Gamma_{\cap}^{C}(D_{2n})) = \frac{n}{2}+4$. □

The girth of the cyclic intersection graph of subgroups of the dihedral groups is given by Proposition 3.14.

Proposition 3.14. *Let D_{2n} be the dihedral group of order $2n$. Then the girth of the cyclic intersection graph of subgroups on the dihedral groups can be written as:*

$$girth(\Gamma_{\cap}^{C}(D_{2n})) = \begin{cases} \infty & if \quad n = p \quad where \ p \ is \ a \ prime \ number, \\ 3 & \quad otherwise. \end{cases}$$

Proof. If $n = p$, then by Theorem 3.3, $\Gamma_{\cap}^{C}(D_{2p}) = K_{1,p+1}$ is the star graph. Specifically, the graph contains no cycle. Hence, $girth(\Gamma_{\cap}^{C}(D_{2p})) = \infty$. If $n \neq p$, then $\Gamma_{\cap}^{C}(D_{2p^r})$, $\Gamma_{\cap}^{C}(D_{2pq})$, $\Gamma_{\cap}^{C}(D_{2(2^r q)})$, $\Gamma_{\cap}^{C}(D_{2p^r q})$ and $\Gamma_{\cap}^{C}(D_{2pqh})$ have subgraphs isomorphic to $K_{r+p^{r-1}}$, K_{p^r+q+1}, $K_{\beta+\gamma}$, K_{p^r+q+r} and $K_{\frac{n}{2}+4}$, respectively. Each of the above subgraphs contain the vertices $\{\langle a\rangle, \langle a^t\rangle, \langle a^t, a^i b\rangle\}$, $1 \leq t < n$ and $0 \leq i \leq t-1$ where $t \mid n$. Then by the vertex adjacency rule $\{\langle a\rangle, \langle a^t\rangle, \langle a^t, a^i b\rangle\}$ is a triangle clique. Consequently, $girth(\Gamma_{\cap}^{C}(D_{2n})) = 3$. □

The diameter of the cyclic intersection graph of subgroups of the dihedral groups is given by Proposition 3.15.

Proposition 3.15. Let D_{2n} be the dihedral group of order $2n$. Then

$$diam(\Gamma_{\cap}^{C}(D_{2n})) = 2, \ for \ all \ n.$$

Proof. If $n = p$, then by Theorem 3.3, $\Gamma_{\cap}^{C}(D_{2p}) = K_{1,p+1}$ is the star graph. Explicitly, the eccentricity of all the vertices is 2. Thus, $diam(\Gamma_{\cap}^{C}(D_{2n})) = 2$. If $n \neq p$, then by Theorem 3.2, D_{2n} is adjacent to all the cyclic subgroups. Select an arbitrary $x_1, x_2 \in \Gamma_{\cap}^{C}(D_{2n})$ such that $x_1 \in \langle a^t, a^i b\rangle$ is a non-cyclic subgroup. However, $x_2 \in \langle a^t\rangle$ is a cyclic subgroup, then $\langle a^t, a^i b\rangle \cap \langle a^t\rangle = \langle a^t\rangle$. Accordingly, the distance $d(x_1, x_2) = d(x_2, x_1) = 1$. Since $x_1 \nsim D_{2n}$ and $x_1 \sim x_2 \sim D_{2n}$, then $d(x_1, \ D_{2n}) = 2$. In addition, the maximum eccentricity is $max(ecc(\Gamma_{\cap}^{C}(D_{2n}))) = 2$. Therefore, $diam(\Gamma_{\cap}^{C}(D_{2n})) = 2$. □

In the following propositions, the chromatic numbers, $\chi(\Gamma_{\cap}^{C}(D_{2n}))$, of the cyclic intersection graph of subgroups of the dihedral groups D_{2n} is obtained.

Proposition 3.16. *Let D_{2n} be the dihedral group of order $2n$. Then the chromatic number, $\chi(\Gamma_{\cap}^{C})$, of the cyclic intersection graph of subgroups of the dihedral groups D_{2n} for $n = p$ and $n = p^r$ takes the form:*

$$\chi(\Gamma_{\cap}^{C}(D_{2n})) = \begin{cases} 2 & if \quad n = p, \\ r + p^{r-1} & if \quad n = p^r, \end{cases}$$

where p is a prime number, and $r > 1$ for $n \in \mathbb{N}$.

Proof. If $n = p$, then by Theorem 3.3, $\Gamma_{\cap}^{C}(D_{2n}))$ is a star graph. Thus, the chromatic number of this graph can be written as $\chi(\Gamma_{\cap}^{C}(D_{2n})) = 2$. Conversely, according to Proposition 3.12, the size of the maximum clique in $\Gamma_{\cap}^{C}(D_{2n})$, when $n = p^r$ and $r > 1 \in \mathbb{N}$ becomes $r + p^{r-1}$. In addition, for proper coloring, each vertex must be assigned to a distinct color. Thus, in this case $\chi(\Gamma_{\cap}^{C}(D_{2n})) = r + p^{r-1}$. □

Proposition 3.17. *Let D_{2n} be the dihedral group of order $2n$. Then the chromatic number ,$\chi(\Gamma_{\cap}^{C})$, of the cyclic intersection graph of subgroups of the dihedral groups D_{2n} for $n = p^r q$ is given by:*

$$\chi(\Gamma_{\cap}^{C}(D_{2n})) = \begin{cases} p^r + q + 1 & if \quad n = pq, \\ \beta + \gamma & if \quad n = 2^r q, \\ p^r + q + r & if \quad n = p^r q, \ p \neq 2, \end{cases}$$

where p and q are prime numbers and $r > 1$ for $r \in \mathbb{N}$.

Proof. Suppose D_{2n} is a dihedral group of order $2n$, then by Proposition 3.13, $\omega(\Gamma_{\cap}^{C}(D_{2n})) = p^r + q + 1$, $\omega(\Gamma_{\cap}^{C}(D_{2n})) = \beta + \gamma$ and $\omega(\Gamma_{\cap}^{C}(D_{2n})) = p^r + q + r$ if $n = pq$, $n = 2^r q$ and $n = p^r q$, respectively. Thus, each vertex must be assigned to a distinct color. Hence $\chi(\Gamma_{\cap}^{C}(D_{2n})) = p^r + q + 1$, $\chi(\Gamma_{\cap}^{C}(D_{2n})) = \beta + \gamma$ and $\chi(\Gamma_{\cap}^{C}(D_{2n})) = p^r + q + r$. □

Proposition 3.18. *Let D_{2n} be the dihedral group of order $2n$, for $n = pqh$ where p, q and h are the prime numbers and p is the smallest prime number between them. Then $\chi(\Gamma_{\cap}^{C}(D_{2n})) = \frac{n}{p} + 4$.*

Proof. Let D_{2n} is a dihedral group of order $2n$. Then according to Proposition 3.13, $\omega(\Gamma_{\cap}^{C}(D_{2n})) = \frac{n}{p} + 4$ if $n = n = pqh$. Thus, each vertex must be assigned to a distinct color. Hence, $\chi(\Gamma_{\cap}^{C}(D_{2n})) = \frac{n}{p} + 4$. □

The edge chromatic numbers, $\chi^e(\Gamma_{\cap}^{C})$, for the cyclic intersection graph of subgroups of the dihedral groups D_{2n} is given as follows proposition.

Proposition 3.19. *Let D_{2n} be the dihedral group of order $2n$. Then the edge chromatic number of the dihedral group is written as:*

$$\chi^e(\Gamma_{\cap}^{C}(D_{2n})) = \begin{cases} n+1 & if \quad n = p, \\ n+d & if \quad n = p^{\,r} \quad or \quad n = pq, \\ d + \Sigma t & if \quad n = p^r q \quad or \quad n = pqh \end{cases}$$

where d is the number of positive divisors of n and Σt is the sum of the non-trivial proper divisors of n.

Proof. Let D_{2n} is the dihedral group of order $2n$. Assume that, $\langle a^t \rangle$, $\langle a^i b \rangle$ and $\langle a^t, a^j b \rangle$ where $t \neq n \;\; t \mid n,\;\; 0 \leq i \leq n,\;\; 1 \leq t \leq n$, and $0 \leq j \leq t-1$ be the subgroups generated by the elements of rotation, reflection as well as combination of the rotations and reflections, respectively. Then $\langle a^t \rangle \cap D_{2n} = \langle a^t \rangle$, $\langle a^i b \rangle \cap D_{2n} = \langle a^i b \rangle$ are all cyclic subgroups, indicating that the edges between D_{2n} and $\langle a^t \rangle, \langle a^i b \rangle$ exist. However, $\langle a^t, a^j b \rangle \cap D_{2n} = \langle a^t, a^j b \rangle$ is a non-cyclic group. Then by definition there is no edge linking $\langle a^t, a^j b \rangle$ and D_{2n}. Consequently, the color for the edges $\langle a^t, a^j b \rangle$ can be used to color D_{2n}. Nevertheless, each edge of the vertex with maximum degree must have a distinct color. As a result, there is a need of $(\Gamma_{\cap}^{C}(D_{2n}))$ colors to color the entire edges of the graph. In short, $\chi^e(\Gamma_{\cap}^{C}(D_{2n})) = \Delta(\Gamma_{\cap}^{C}(D_{2n}))$. Then by Proposition 3.11, $\chi^e(\Gamma_{\cap}^{C}(D_{2p})) = \chi^e(\Gamma_{\cap}^{C}(D_{2pq})) = n + 1$, $\chi^e(\Gamma_{\cap}^{C}(D_{2p^r})) = \chi^e(\Gamma_{\cap}^{C}(D_{2pq})) = n + d$ and $\chi^e(\Gamma_{\cap}^{C}(D_{2p^r q})) = \chi^e(\Gamma_{\cap}^{C}(D_{2pqh})) = d + \Sigma t$ □

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