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The Square Root of Nonsingular Matrices with Non-negative Eigenvalues

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Abstract

In this paper, we introduce a method to find the square root of non-singular matrices with non-negative eigenvalues. Our method extends and generalizes the corresponding approach for 2×2 matrix. In addition, an illustrative example is given.

1 Introduction

Let M_n denote the set of $n \times n$ complex matrices. The identity in M_n is denoted by I_n. A matrix B is a square root of a matrix A if $B^2 = A$ and we will denote it by $B = \sqrt{A}$. The square root matrix with most practical interest is the one whose eigenvalues are non-negative, which is called the principal square root. If A is nonsingular and has non-negative eigenvalues, then A has a unique principal square root.

The square root of a matrix has applications in many problems like computation of the polar decomposition, solution to differential equations, the

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matrix sign function, and Markov models of finance. In 2011, Al-Tamimi [4] introduced a new method for finding the square root of a certain class of 2×2 matrix using the Cayley-Hamilton theorem. Our method generalizes that method for $n \times n$ positive semidefinte matrices.

2 Main Results

In what follows, the set of all $n \times n$ non-singular matrices $M'_n(\mathbb{R})$ over $\mathbb{R}$ with non-negative eigenvalues. Let $\chi_A(x)$ denote the characteristic polynomial of a matrix A. If λ is an eigenvalue of matrix A, then $\chi_A(\lambda) = 0$.

Lemma 2.1. *Let $A \in M'_n(\mathbb{R})$. If λ is an eigenvalue of A, then neither $\sqrt{\lambda}$ nor $-\sqrt{\lambda}$ is an eigenvalue of matrix $B = \sqrt{A}$.*

Proof. Let λ be an eigenvalue of A. Then $\lambda \geq 0$ and $Av = \lambda v$ for some non-zero vector v. Thus $B^2 v = \lambda v$. So neither $\sqrt{\lambda}$ nor $-\sqrt{\lambda}$ is an eigenvalue of B.

□

Proposition 2.2. *Let $A \in M'_n(\mathbb{R})$.*
If $B = \sqrt{A}$, then $\chi_A(x) = (-1)^n \chi_B(\sqrt{x})\chi_B(-\sqrt{x})$.

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_k, \lambda_{k+1}, \ldots, \lambda_n$ be all eigenvalues of A. Using Lemma 2.1, suppose that $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_k}$ and $-\sqrt{\lambda_{k+1}}, -\sqrt{\lambda_{k+2}}, \ldots, -\sqrt{\lambda_n}$ are all the eigenvalues of B. Then $\chi_B(x) = (x-\sqrt{\lambda_1}) \cdots (x-\sqrt{\lambda_k})(x+\sqrt{\lambda_{k+1}}) \cdots (x+\sqrt{\lambda_n})$ and

$$\begin{aligned}
\chi_B(\sqrt{x})\chi_B(-\sqrt{x}) &= (\sqrt{x} - \sqrt{\lambda_1})(\sqrt{x} - \sqrt{\lambda_2}) \cdots (\sqrt{x} - \sqrt{\lambda_k}) \\
&\quad (\sqrt{x} + \sqrt{\lambda_{k+1}}) \cdots (\sqrt{x} + \sqrt{\lambda_{n-1}})(\sqrt{x} + \sqrt{\lambda_n}) \\
&\quad (-\sqrt{x} - \sqrt{\lambda_1})(-\sqrt{x} - \sqrt{\lambda_2}) \cdots (-\sqrt{x} - \sqrt{\lambda_k}) \\
&\quad (-\sqrt{x} + \sqrt{\lambda_{k+1}}) \cdots (-\sqrt{x} + \sqrt{\lambda_{n-1}})(-\sqrt{x} + \sqrt{\lambda_n}) \\
&= (\lambda_1 - x)(\lambda_2 - x) \cdots (\lambda_k - x)(\lambda_{k+1} - x) \cdots (\lambda_n - x) \\
&= (-1)^n (x - \lambda_1) \cdots (x - \lambda_k)(x - \lambda_{k+1}) \cdots (x - \lambda_n) \\
&= (-1)^n \chi_A(x).
\end{aligned}$$

□

Theorem 2.3. *Let $A \in M'_n(\mathbb{R})$ and let $B = \sqrt{A}$.*

(1). If $n = 2k$, then $B = (A^k + b_{2k-2}A^{k-1} + \cdots + b_0 I)(-b_{2k-1}A^{k-1} - \cdots - b_1 I)^{-1}$.

(2). If $n = 2k+1$, then $B = (-b_{2k}A^k - \cdots - b_0 I)(A^k + b_{2k-1}A^{k-1} + \cdots + b_1)^{-1}$.

Proof. Let $\chi_A(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ and let $\chi_B(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$ be characteristic polynomials of A and B, respectively. By the Cayley-Hamilton theorem, we have $\chi_B(B) = 0$.
(1). If $n = 2k$, then

$$B^{2k} + b_{2k-1}B^{2k-1} + \cdots + b_1 B + b_0 I = 0$$

$$B^{2k} + b_{2k-2}B^{2k-2} + \cdots + b_0 I = -b_{2k-1}B^{2k-1} - \cdots - b_1 B$$

$$B^{2k} + b_{2k-2}B^{2k-2} + \cdots + b_0 I = B(-b_{2k-1}B^{2k-2} - \cdots - b_1 I).$$

Thus $B = (B^{2k} + b_{2k-2}B^{2k-2} + \cdots + b_0 I)(-b_{2k-1}B^{2k-2} - \cdots - b_1 I)^{-1}$ implies that $B = (A^k + b_{2k-2}A^{k-1} + \cdots + b_0 I)(-b_{2k-1}A^{k-1} - \cdots - b_1 I)^{-1}$.
(2). If $n = 2k+1$, then similarly by the Cayley-Hamilton theorem we get

$$B(B^{2k} + b_{2k-1}B^{2k-2} + \cdots + b_1 I) = -b_{2k}B^{2k} - \cdots - b_0 I.$$

So $B = (-b_{2k}B^{2k} - \cdots - b_0 I)(B^{2k} + b_{2k-1}B^{2k-2} + \cdots + b_1 I)^{-1}$. Consequently,

$$B = (-b_{2k}A^k - \cdots - b_0 I)(A^k + b_{2k-1}A^{k-1} + \cdots + b_1 I)^{-1}.$$

□

The following example illustrates our method.

Example 2.4. *Let* $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & 9 \end{bmatrix}$. *Then we have to find $\sqrt{A}$ by using Theorem 2.3.*

The characteristic polynomial of A is $\chi_A(\lambda) = x^3 - 14x^2 + 49x - 36$. If

$B^2 = A$, then, by Proposition 2.2, we have

$$\begin{aligned}(-1)^n(P_A(x)) &= (P_B(\sqrt{x}))(P_B(-\sqrt{x}))\\ (-1)^3(x^3-14x^2+49x-36) &= ((-\sqrt{x})^3+b_2(\sqrt{x})^2-b_1(\sqrt{x})+b_0)\\ &\quad ((\sqrt{x})^3+b_2(\sqrt{x})^2+b_1(\sqrt{x})+b_0)\\ &= -x^3-b_2(\sqrt{x})^5-b_1(\sqrt{x})^4+b_0(\sqrt{x})^3\\ &\quad +b_2(\sqrt{x})^5+b_2b_2(\sqrt{x})^4+b_2b_1(\sqrt{x})^3+b_2b_0(\sqrt{x})^2\\ &\quad -b_1(\sqrt{x})^4-b_1b_2(\sqrt{x})^3-b_1b_1(\sqrt{x})^2-b_1b_0(\sqrt{x})\\ &\quad +b_0(\sqrt{x})^3+b_0b_2(\sqrt{x})^2+b_0b_1(\sqrt{x})+b_0b_0\\ &= -x^3-(b_1-b_2b_2+b_1)x^2-(-b_2b_0+b_1b_1-b_0b_2)x-(-b_0b_0)\\ &= -x^3-(2b_1-b_2^2)x^2-(-2b_2b_0+b_1^2)x-(-b_0^2)\\ &= (-1)(x^3+(2b_1-b_2^2)x^2+(-2b_2b_0+b_1^2)x+(-b_0^2)).\end{aligned}$$

Thus $b_0^2 = 36$, $-2b_2b_0+b_1^2 = 49$ and $-14 = 2b_1-b_2^2$. By direct computation, we get $(b_0, b_1, b_2) = (6, -7, 0), (-6, -7, 0), (-6, 1, 4), (6, 1, -4), (-6, 11, -6), (6, 11, 6), (-6, -5, 2), (6, -5, -2)$. If we consider $(b_0, b_1, b_2) = (6, -7, 0)$, then by Theorem 2.3

$$\begin{aligned} B &= (b_2A+b_0I)(-A-b_1I)^{-1}\\ &= \left(0+6\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right)\left(-\begin{bmatrix}1&1&1\\0&4&1\\0&0&9\end{bmatrix}-(-7)\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right)^{-1}\\ &= \begin{bmatrix}1&\frac{1}{3}&-\frac{2}{3}\\0&2&-1\\0&0&-3\end{bmatrix}.\end{aligned}$$

In the same way, we have

$$\begin{bmatrix}-1&-\frac{1}{3}&\frac{2}{3}\\0&-2&1\\0&0&3\end{bmatrix}, \begin{bmatrix}1&-1&-\frac{2}{5}\\0&-2&-\frac{1}{5}\\0&0&-3\end{bmatrix}, \begin{bmatrix}-1&1&\frac{2}{5}\\0&2&\frac{1}{5}\\0&0&3\end{bmatrix},$$

$$\begin{bmatrix}1&\frac{1}{3}&\frac{7}{30}\\0&2&\frac{1}{5}\\0&0&3\end{bmatrix}, \begin{bmatrix}-1&-\frac{1}{3}&-\frac{7}{30}\\0&-2&-\frac{1}{5}\\0&0&-3\end{bmatrix}, \begin{bmatrix}1&-1&\frac{1}{2}\\0&-2&1\\0&0&3\end{bmatrix}, \text{ and } \begin{bmatrix}-1&1&-\frac{1}{2}\\0&2&-1\\0&0&-3\end{bmatrix}$$

are square roots of A.

References

[1] A. Bjorck, S. Hammarling, A Schur method for the Square Root of a Matrix, *Linear Algebra Appl.,* **52/53,** (1983), 127–140.

[2] Charles R. Johnson, Kazuyoshi Okuba, Robert Reams, Uniqueness of Matrix Square Roots and an Application, *Linear Algebra and its applications,* **323,** (2001), 51–60.

[3] Donald Sullivan, The Square Roots of 2×2 Matrices, *Mathematics Magazine,* **66,** no. 5, (1993), 314–316.

[4] Ihab Ahmad Abd Al-Baset Al-Tamimi, The Square Roots of $2x2$ Invertible Matrices, *International Journal of Difference Equations*, **6**, no. 1, (2011), 61–64.

[5] Nicholas J. Higham, Newton's Method for the Matrix Square Root, *Math. of Computation*, **46,** (1996), 537–549.