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Generalized Continuous Functions in Bigeneralized Topological Spaces

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Abstract

In this paper, we characterize the generalized continuous functions in bigeneralized topological spaces.

1 Introduction

The idea of bigeneralized topological space was first introduced by Boonpok [5] in 2011. On the other hand, Benchalli et al. [3] introduced the generalized star $\omega\alpha$-sets (briefly $g^*\omega\alpha$-sets) in topological spaces in 2015.

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In this paper, we introduce the generalized star continuity of functions in bigeneralized topological spaces as an extension of the work of Benchalli et al. [3]. Moreover, we explore some properties and characterizations of this topological concept.

For standard terminologies and notations in topology, the readers may refer to [6]. Let X be a nonempty set. A subset μ of $\mathscr{P}(X)$ is said to be a generalized topology (briefly GT) on X if $\varnothing \in \mu$ and the arbitrary union of elements of μ belongs to μ.

If μ is a GT on X, then (X, μ) is said to be a *generalized topological space* (briefly GTS), and the elements of μ are called *μ-open sets*. The complement of a μ-open set is called *μ-closed set*. If $A \subseteq X$, then the *μ-closure of* A, denoted by $c_\mu(A)$, is the intersection of all μ-closed sets containing A. The *μ-interior of* A, denoted by $i_\mu(A)$, is the union of all μ-open sets contained in A.

The following definitions were introduced by Benchalli et al. [1] in 2009. A set A of a GTS (X, μ) is said to be μ-α-closed if $c_\mu(i_\mu(c_\mu(A)) \subseteq A$ and μ-$\omega\alpha$-closed if $\alpha c_\mu(A) \subseteq U$ whenever $A \subseteq U$ is μ-ω-open in X. The complement of a μ-$\omega\alpha$-closed set is *μ-$\omega\alpha$-open* set.

A subset A of X is said to be *μ-generalized star $\omega\alpha$-closed* (briefly μ-$g^*\omega\alpha$-closed) set if $c_\mu(A) \subseteq U$ whenever $A \subseteq U$ and U is μ-$\omega\alpha$-open in X. The complement of μ-$g^*\omega\alpha$-closed set is said to be *μ-$g^*\omega\alpha$-open set*. If A is both μ-$g^*\omega\alpha$-closed set and μ-$g^*\omega\alpha$-open set, then A is said to be *μ-$g^*\omega\alpha$-clopen set*. The union of all the μ-$g^*\omega\alpha$-open sets contained in A is called the *μ-$g^*\omega\alpha$-interior of* A, denoted by $g^*\omega\alpha i_\mu(A)$. The intersection of all the μ-$g^*\omega\alpha$-closed sets containing A is called the *μ-$g^*\omega\alpha$-closure of* A denoted by $g^*\omega\alpha c_\mu(A)$.

If μ_1 and μ_2 are generalized topologies on X, then the triple (X, μ_1, μ_2) is said to be a *bigeneralized topological space* (briefly BGTS). Throughout this paper, m and n take values from the set $\{1, 2\}$ where $m \neq n$.

The following definition is due to Boonpok et al. [4].

Definition 1.1. [4] Let $f : (X, \mu_X^1, \mu_X^2) \to (Y, \mu_Y^1, \mu_Y^2)$ be a function. Then f is *$\mu^{(m,n)}$-continuous at a point* $x \in X$ if for each μ_Y^m-open set V containing $f(x)$, there exists a μ_X^n-open set U containing x such that $f(U) \subseteq V$. If f is $\mu^{(m,n)}$-continuous at every point $x \in X$, then f is *$\mu^{(m,n)}$-continuous*.

2 Main Results

In this section, we introduce different forms of $\mu^{(m,n)}$-$g^*\omega\alpha$ continuous functions in a BGTS, investigate some of their properties, and establish their relationships. Finally, we characterize the generalized star continuous functions in a bigeneralized topological space.

Definition 2.1. A function $f : (X, \mu_X^1, \mu_X^2) \to (Y, \mu_Y^1, \mu_Y^2)$ is said to be:

(i) $\mu^{(m,n)}$-$g^*\omega\alpha$ *continuous at a point* $x \in X$ if for each μ_Y^m-open set V containing $f(x)$, there exists a μ_X^n-$g^*\omega\alpha$ open set U containing x such that $f(U) \subseteq V$.

(ii) $\mu^{(m,n)}$-$g^*\omega\alpha$ *continuous* if f is $\mu^{(m,n)}$-$g^*\omega\alpha$ continuous at every point $x \in X$.

(iii) *pairwise* μ-$g^*\omega\alpha$ *continuous* if f is $\mu^{(1,2)}$-$g^*\omega\alpha$ continuous and $\mu^{(2,1)}$-$g^*\omega\alpha$ continuous.

Lemma 2.2. Every μ-closed set is μ-$g^*\omega\alpha$-closed.

The next corollary is immediate from Lemma 2.2.

Corollary 2.3. Every μ-open set is μ-$g^*\omega\alpha$-open.

The next result establishes a relationship between continuity and generalized star continuity in a bigeneralized topological space.

Theorem 2.4. Every $\mu^{(m,n)}$-continuous function is $\mu^{(m,n)}$-$g^*\omega\alpha$-continuous.

Proof. Let $x \in X$. Since f is $\mu^{(m,n)}$-continuous function, by Definition 1.1, for μ_Y^m-open set V containing $f(x)$, there exists a μ_X^n-open set U containing x such that $f(U) \subseteq V$. By Corollary 2.3, there exists a μ_X^n-$g^*\omega\alpha$ open set U containing x such that $f(U) \subseteq V$. Therefore, the conclusion holds. □

The following lemma establishes the interior and closure properties with respect to the generalized star open sets.

Lemma 2.5. Let (X, μ) be a GTS and A, B and F be subsets of X.

(*i*) If A is μ-$g^*\omega\alpha$-open, then $A = g^*\omega\alpha i_\mu(A) = g^*\omega\alpha i_\mu\ (g^*\omega\alpha i_\mu(A))$;

(*ii*) $x \in g^*\omega\alpha i_\mu(A)$ if and only if there exist a μ-$g^*\omega\alpha$-open set U with $x \in U \subseteq A$; and (*iii*) If $A \subseteq B$, then $g^*\omega\alpha i_\mu(A) \subseteq g^*\omega\alpha i_\mu(B)$.

(*iii*) $y \in g^*\omega\alpha c_\mu(A)$ if and only if for every μ-$g^*\omega\alpha$ open set U with $y \in U$, $U \cap A \neq \varnothing$;

The following result characterizes the generalized star continuous functions in bigeneralized topological space.

Theorem 2.6. For a function $f : (X, \mu_X^1, \mu_X^2) \to (Y, \mu_Y^1, \mu_Y^2)$, the following properties are equivalent:

(*i*) f is $\mu^{(m,n)}$-$g^*\omega\alpha$ continuous at $a \in X$;

(*ii*) $x \in g^*\omega\alpha i_{\mu_X^n}(f^{-1}(V))$ for every $V \in \mu_Y^m$ containing $f(x)$;

(*iii*) $x \in g^*\omega\alpha i_{\mu_X^n}(f^{-1}(B))$ for every $B \subseteq Y$ with $x \in f^{-1}(i_{\mu_Y^m}(B))$;

(*iv*) $x \in f^{-1}(F)$ for every μ_Y^m-closed subset F of Y such that $x \in g^*\omega\alpha c_{\mu_X^n}(f^{-1}(F))$

Proof. Let $f : X \to Y$ be a function and let $x \in X$.
(*i*) $\Leftrightarrow$ (*ii*): Let $V \in \mu_Y^m$ containing $f(x)$. Since f is $\mu^{(m,n)}$-$g^*\omega\alpha$ continuous at x, there exists a μ_X^n-$g^*\omega\alpha$ open set U containing x such that $f(U) \subseteq V$. Hence, $x \in U \subseteq f^{-1}(V)$. This implies that $x \in g^*\omega\alpha i_{\mu_X^n}(f^{-1}(V))$.

Conversely, let $V \in \mu_Y^m$ with $f(x) \in V$. By (ii), $x \in g^*\omega\alpha i_{\mu_X^n}(f^{-1}(V))$. Hence, there exists a μ_X^n-$g^*\omega\alpha$ open set U with $x \in U \subseteq f^{-1}(V)$. Thus, $f(U) \subseteq V$. Therefore, f is $\mu^{(m,n)}$-$g^*\omega\alpha$ continuous at $x \in X$.

(*ii*) $\Rightarrow$ (*iii*): Let $B \subseteq Y$ with $x \in f^{-1}(i_{\mu_Y^m}(B))$. Then $f(x) \in i_{\mu_Y^m}(B)$. Since $i_{\mu_Y^m}(B) \in \mu_Y^m$, by (ii) we have, $x \in g^*\omega\alpha i_{\mu_X^n}(f^{-1}(i_{\mu_Y^m}B)) \subseteq g^*\omega\alpha i_{\mu_X^n}(f^{-1}(B)$. Thus, $x \in g^*\omega\alpha i_{\mu_X^n}(f^{-1}(B))$.

(*iii*) $\Rightarrow$ (*iv*): Let F be a μ_Y^m-closed subset of Y such that $x \notin f^{-1}(F)$. Then $x \in X \setminus f^{-1}(F) = f^{-1}(Y \setminus F) = f^{-1}(i_{\mu_Y^m}(Y \setminus F))$ since $Y \setminus F$ is μ_Y^m open. By (iii), $x \in g^*\omega\alpha i_{\mu_X^n}(f^{-1}(Y \setminus F)) = g^*\omega\alpha i_{\mu_X^n}(X \setminus f^{-1}(F)) = X \setminus g^*\omega\alpha c_{\mu_X^n}(f^{-1}(F))$. Hence, $x \notin g^*\omega\alpha c_{\mu_X^n}(f^{-1}(F)$.

(*iv*) $\Rightarrow$ (*ii*): Let $V \in \mu_Y^m$ with $f(x) \in V$. Suppose that $x \notin g^*\omega\alpha i_{\mu_X^n}(f^{-1}(V))$. Then $x \in X \setminus g^*\omega\alpha i_{\mu_X^n}(f^{-1}(V) = g^*\omega\alpha c_{\mu_X^n}(X \setminus f^{-1}(V)) = g^*\omega\alpha c_{\mu_X^n}(f^{-1}(Y \setminus V))$. By (iv), $x \in f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$. This implies that $x \notin f^{-1}(V)$ which is a contradiction since $f(x) \in V$. Therefore, $x \in g^*\omega\alpha i_{\mu_X^n}(f^{-1}(V))$. $\square$

The following result gives a sufficient condition for a function to be generalized star continuous in a bigeneralized topological space.

Theorem 2.7. For a function $f : (X, \mu_X^1, \mu_X^2) \to (Y, \mu_Y^1, \mu_Y^2)$, the following properties are equivalent:

(*i*) f is $\mu^{(m,n)}$-$g^*\omega\alpha$ continuous;

(*ii*) $f^{-1}(V) = g^*\omega\alpha i_{\mu_X^n}(f^{-1}(V))$ for every $V \in \mu_Y^m$;

(*iii*) $f^{-1}(i_{\mu_Y^m}(B)) \subseteq g^*\omega\alpha i_{\mu_X^n}(f^{-1}(B))$ for every $B \subseteq Y$;

(*iv*) $g^*\omega\alpha c_{\mu_X^n}(f^{-1}(F)) = f^{-1}(F)$ for every μ_Y^m-closed subset F of Y.

Proof. Let $f : X \to Y$ be a function and let $x \in X$.
$(i) \Rightarrow (ii)$: Let $V \in \mu_Y^m$ and $x \in f^{-1}(V)$. Then $f(x) \in V$. By Theorem 2.7 (ii), $x \in i_{\mu_X^n}(f^{-1}(V))$. Since $g^*\omega\alpha i_{\mu_X^n}(f^{-1}(V)) \subseteq f^{-1}(V)$, we have $f^{-1}(V) = g^*\omega\alpha i_{\mu_X^n}(f^{-1}(V))$.

$(ii) \Rightarrow (iii)$: Let $B \subseteq Y$. Since $i_{\mu_Y^m}(B) \in \mu_Y^m$, by (ii) we have $f^{-1}(i_{\mu_Y^m}(B)) = g^*\omega\alpha i_{\mu_X^n}(f^{-1}(i_{\mu_Y^m}(B)) \subseteq g^*\omega\alpha i_{\mu_X^n}(f^{-1}(B))$. Thus, $f^{-1}(i_{\mu_Y^m}(B)) \subseteq g^*\omega\alpha i_{\mu_X^n}(f^{-1}(B))$.

$(iii) \Rightarrow (iv)$: Let F be a μ_Y^m-closed subset of Y. Then by (iii), $f^{-1}(Y \setminus F)$ $= f^{-1}(i_{\mu_Y^m}(Y \setminus F)) \subseteq g^*\omega\alpha i_{\mu_X^n}(f^{-1}(Y \setminus F)) = g^*\omega\alpha i_{\mu_X^n}(X \setminus f^{-1}(F)) = X \setminus g^*\omega\alpha c_{\mu_X^n}(f^{-1}(F))$. Thus, $g^*\omega\alpha c_{\mu_X^n}(f^{-1}(F)) \subseteq f^{-1}(F)$. Hence, $g^*\omega\alpha c_{\mu_X^n}(f^{-1}(F))$ $= f^{-1}(F)$.

$(iv) \Rightarrow (i)$: Let $x \in X$ and F be a μ_Y^m-closed subset of Y with $x \in g^*\omega\alpha c_{\mu_X^n}(f^{-1}(F)$. By (iv), $x \in f^{-1}(F)$. Thus by Theorem 2.7 (iv), f is $\mu^{(m,n)}$-$g^*\omega\alpha$ continuous at x. Since x is arbitrary, f is $\mu^{(m,n)}$-$g^*\omega\alpha$ continuous. □

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