International Journal of Mathematics and
Computer Science, **18**(2023), no. 4, 627–636

# Connection between the Hermite and Gegenbauer polynomials using the Lewanowicz method

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(Received October 28, 2022, Revised November 20, 2022,
Accepted February 21, 2023, Published May 31, 2023)

**Abstract**

In this article, we use the Lewanowicz method with the Fields-Wimp formula to solve the connection problem between continuous classical orthogonal polynomials of Gegenbauer and Hermite and the connection formulas between shifted Laguerre and Jacobi polynomials, and the known connection coefficients between the Gegenbauer polynomials and the Laguerre polynomials are established.

## 1 Introduction

The connection problem is to find the coefficients $c_{nk}$ in the expansion of a polynomial $P_n(x)$ in terms of an arbitrary sequence of orthogonal polynomials

**Keywords and phrases:** Orthogonal polynomials, Gegenbauer polynomials, Hermite polynomials, Connection problems, hypergeometric functions.
**AMS (MOS) Subject Classifications**: 33D45, 33C45.
**ISSN** 1814-0432, 2023, http://ijmcs.future-in-tech.net

$\{Q_k(x)\}$ :

$$P_n(x) = \sum_{k=0}^{n} c_{nk} Q_k(x). \tag{1.1}$$

A wide variety of methods have been devised for computing the connection coefficients $c_{nk}$, either in closed form or by means of recurrence relations, usually in $k$. Lewanowicz [1] has shown that the connection problem (1.1) can, sometimes be solved by taking advantage of known results from the theory of generalized hypergeometric functions, derived by Fields and Wimp [2].

The hypergeometric functions method has been devised to solve the connection problem, that involving classical orthogonal polynomials . [1, 3, 4, 5, 6]. These methods have also been used by Sánchez-Ruiz [7] to obtain the connection formulae involving squares of Gegenbauer Polynomials. In [8], authors seeking for solving (1.1) for a much wider class of polynomials, defined by terminating hypergeometric series, obtain connection formulae for Wilson and Racah polynomials with special parameter values. They also solve the connection problem for the families of generalized Jacobi and Laguerre polynomials defined by Sister Celine.

Linearization and connection problems arise in the calculation of information entropies of quantum systems in position and momentum spaces [9]. Other examples of applications of linearization and connection problems to quantum physics include transformation formulas between wave functions in different coordinate systems [10], inter-basis expansions for potentials of equal [11] and different [12] dimensionality, Talmi-Brody-Moshinsky coefficients in nuclear structure [13], or two-center, two/three-electron integrals in variational atomic analysis [14].

While in many of these applications we are interested in finding either the explicit form of the linearization/connection coefficients or recurrence relations for them. In the harmonic analysis with respect to sequences of orthogonal polynomials, it is important to know the sign properties, especially if they are positive or non-negative [15]. Thus, for example, the non-negativity of certain connection coefficients for Gegenbauer polynomials played a key role in de Branges' proof [16] of the Bieberbach conjecture in complex analysis, and recently sign properties of connection coefficients have been used to study the behavior of polynomial zeros [17].

## 2 Notation and preliminary results

The generalized hypergeometric function is defined by

$$ {}_pF_q\left( \begin{array}{c} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{array} \middle| x \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{x^k}{k!}, \tag{2.2} $$

where $(a)_n$ represents the Pochhammer symbol and it used in the theory of special functions to represent the rising factorial.

$$ (a)_n := a(a+1)(a+2)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)} = \frac{(-1)^n\,\Gamma(1-a)}{\Gamma(1-a-n)}, $$

$$ (a)_0 = 1, $$

where $a_i \in C$, $1 \leq i \leq p$, $b_j \in C$, $1 \leq j \leq q$, with $b_j \notin N_0$. Throughout this article, the letters $p, q, r, s, t, u$ and $n$ stand for nonnegative integers. We call $x$ the argument of the function, and $a_j$, $b_j$ the parameters. To shorten the notation for the left-hand side of (2.2), we will write it as

$$ {}_pF_q\left( \begin{array}{c} [a_p] \\ [b_q] \end{array} \middle| x \right) = \sum_{k=0}^{\infty} \frac{[a_p]_k\, x^k}{[b_q]_k\, k!}, \tag{2.3} $$

where $[a_p]$ and $[b_q]$ represent the sets $\{a_1, a_2, ..., a_p\}$ and $\{b_1, b_2, ..., b_q\}$, respectively. We use the abbreviated notation

$$ [a_p]_k = \prod_{i=1}^{p} (a_i)_k, \qquad [b_q]_k = \prod_{j=1}^{q} (b_j)_k. \tag{2.4} $$

To prove the theorems in section 3, we use known results from the theory of generalized hypergeometric functions, derived by Fields and Wimp (See [2], [3, Vol. II, p. 7]).

$$ \begin{aligned} {}_{p+r+1}F_{q+s}\left( \begin{array}{c} -n, [a_p], [c_r] \\ [b_q], [d_s] \end{array} \middle| zw \right) &= \sum_{k=0}^{n} \binom{n}{k} \frac{[a_p]_k\, [\alpha_t]_k\, z^k}{[b_q]_k [\beta_u]_k (k+\zeta)_k} \\ &\times {}_{p+t+1}F_{q+u+1}\left( \begin{array}{c} k-n, [k+a_p], [k+\alpha_t] \\ 2k+\zeta+1, [k+b_q], [k+\beta_u] \end{array} \middle| z \right) \\ &\times {}_{r+u+2}F_{s+t}\left( \begin{array}{c} -k, k+\zeta, [c_r], [\beta_u] \\ [d_s], [\alpha_t] \end{array} \middle| w \right), \end{aligned} \tag{2.5} $$

$$
\begin{aligned}
&{}_{p+r+1}F_{q+s}\left(\begin{array}{c} -n, [a_p], [c_r] \\ [b_q], [d_s] \end{array} \middle| zw\right) = \sum_{k=0}^{n} \binom{n}{k} \frac{[a_p]_k\, [\alpha_t]_k\, z^k}{[b_q]_k [\beta_u]_k} \\
&\times {}_{p+t+1}F_{q+u}\left(\begin{array}{c} k-n, [k+a_p], [k+\alpha_t] \\ [k+b_q], [k+\beta_u] \end{array} \middle| z\right) \\
&\times {}_{r+u+1}F_{s+t}\left(\begin{array}{c} -k, [c_r], [\beta_u] \\ [d_s], [\alpha_t] \end{array} \middle| w\right). \qquad (2.6)
\end{aligned}
$$

## 2.1 Classical orthogonal polynomials of continuous variable

**Hermite polynomials [18]**

$$H_n(x) = (2x)^n \; {}_2F_0\left(\begin{array}{c} -\frac{n}{2}, \frac{1}{2} - \frac{n}{2} \\ - \end{array} \middle| -\frac{1}{x^2}\right).$$

**Laguerre polynomials [18]**

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!} \; {}_1F_1\left(\begin{array}{c} -n \\ \alpha+1 \end{array} \middle| x\right).$$

**Jacobi polynomials [3, Vol. I, p.274]**

$$
\begin{aligned}
P_n^{(\alpha,\beta)}(x) &= (-1)^n \frac{(\beta+1)_n}{n!} \; {}_2F_1\left(\begin{array}{c} -n, n+\alpha+\beta+1 \\ \beta+1 \end{array} \middle| \frac{1+x}{2}\right). \\
&= \frac{(\alpha+1)_n}{n!} \; {}_2F_1\left(\begin{array}{c} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{array} \middle| \frac{1-x}{2}\right).
\end{aligned}
$$

In many problems it is more convenient to work with shifted Jacobi polynomials, defined as [3, Vol. I, p.273]

$$R_n^{(\alpha,\beta)}(x) := P_n^{(\alpha,\beta)}(2x-1) = (-1)^n \frac{(\beta+1)_n}{n!} \; {}_2F_1\left(\begin{array}{c} -n, n+\alpha+\beta+1 \\ \beta+1 \end{array} \middle| x\right).$$

The *Gegenbauer polynomials* $C_n^{(\alpha)}(x)$ are essentially the symmetric Jacobi polynomials $P_n^{(\alpha,\alpha)}$, and they have the hypergeometric representation [15, p.77].

$$C_n^{(\alpha)}(x) = \frac{(2\alpha)_n}{n!} \; {}_2F_1\left(\begin{array}{c} -\frac{n}{2}, \frac{n}{2}+\alpha \\ \alpha+\frac{1}{2} \end{array} \middle| 1-x^2\right).$$

# 3 Results

In this section, we use the equations (2.5), (2.6) and the different hypergeometric series representation the classical continuous orthogonal polynomials to solve the connection problem among them.

## 3.1 Theorem (Connection formula for Hermite Polynomials in series of Gegenbauer Polynomials)

$$H_N(x) = \sum_{r=0}^{[\frac{N}{2}]} \frac{N!\lambda^{\frac{N}{2}}}{r!(\lambda)_{N-2r}(N-2r+\lambda+1)_r} {}_2F_0\left( \begin{array}{c} -r, -N+r-\lambda \\ - \end{array} \middle| -\frac{1}{\lambda} \right) C^{(\lambda)}_{N-2r}\left(\frac{x}{\sqrt{\lambda}}\right)$$

**Proof:** Some of the hypergeometric representations of Hermite and Gegenbauer polynomials for $\epsilon = 0, 1$ are given by:

$$H_{2n+\epsilon}(x) = (-1)^n\, 2^{2n+\epsilon}\, x^\epsilon \left(\frac{1}{2}+\epsilon\right)_n {}_1F_1\left( \begin{array}{c} -n \\ \frac{1}{2}+\epsilon \end{array} \middle| x^2 \right) \tag{3.7}$$

$$C^{(\lambda)}_{2k+\epsilon}\left(\frac{x}{\sqrt{\lambda}}\right) = \frac{(2\lambda)_{2k+\epsilon}(-1)^k\left(\frac{1}{2}+\epsilon\right)_k}{(\lambda+\frac{1}{2})_k(2k+\epsilon)!} \left(\frac{x}{\sqrt{\lambda}}\right)^\epsilon {}_2F_1\left( \begin{array}{c} -k, k+\lambda+\epsilon \\ \frac{1}{2}+\epsilon \end{array} \middle| \frac{x^2}{\sqrt{\lambda}} \right) \tag{3.8}$$

Using formula (2.6) with the following identification

$$[d_s] = \left\{\frac{1}{2}+\epsilon\right\}, \quad s = 1 \quad t = 0, \quad r = 0, \quad u = 0,$$

$$\omega = \left(\frac{x^2}{\lambda}\right), \quad q = 0, \quad p = 0, \quad z = \lambda, \quad \zeta = \lambda + \epsilon$$

it is obtained

$$ {}_1F_1\left( \begin{array}{c} -n \\ \frac{1}{2}+\epsilon \end{array} \middle| x^2 \right) = \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) \frac{\lambda^k}{(k+\lambda+\epsilon)_k} {}_1F_1\left( \begin{array}{c} k-n \\ 2k+\lambda+\epsilon+1 \end{array} \middle| \lambda \right)$$
$$ {}_2F_1\left( \begin{array}{c} -k, k+\lambda+\epsilon \\ \frac{1}{2}+\epsilon \end{array} \middle| \frac{x^2}{\lambda} \right) \tag{3.9}$$

We multiply (3.9), by convenient terms that to the left of the equality lead to (3.7), and the last part on the right of equality arriving at (3.8); getting

$$\begin{aligned} H_{2n+\epsilon}(x) \quad &= \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) \frac{(-1)^{n-k}2^{2n+\epsilon}\left(\frac{1}{2}+\epsilon\right)_n (\lambda+\frac{1}{2})_k(2k+\epsilon)!\lambda^{k+\frac{\epsilon}{2}}}{(k+\lambda+\epsilon)_k\,(2\lambda)_{2k+\epsilon}\left(\frac{1}{2}+\epsilon\right)_k} \\ &\times {}_1F_1\left( \begin{array}{c} k-n \\ 2k+\lambda+\epsilon+1 \end{array} \middle| \lambda \right) C^{(\lambda)}_{2k+\epsilon}\left(\frac{x}{\sqrt{\lambda}}\right), \end{aligned} \tag{3.10}$$

using the well-known relationship

$$ {}_1F_1\left(\begin{matrix} k-n \\ 2k+\lambda+\epsilon+1 \end{matrix}\middle|\lambda\right) = \frac{(-1)^{n-k}(\lambda)^{n-k}}{(2k+\lambda+\epsilon+1)_{n-k}}\,{}_2F_0\left(\begin{matrix} k-n,-n-k-\lambda-\epsilon \\ - \end{matrix}\middle|-\frac{1}{\lambda}\right), \tag{3.11} $$

we replace this transformation in (3.10)

$$ H_{2n+\epsilon}(x) = \sum_{k=0}^{n} \frac{n!2^{2n+\epsilon}\left(\frac{1}{2}+\epsilon\right)_n (\lambda+\frac{1}{2})_k(2k+\epsilon)!(\lambda)^{n+\frac{\epsilon}{2}}}{(n-k)!k!(k+\lambda+\epsilon)_k\,(2\lambda)_{2k+\epsilon}\left(\frac{1}{2}+\epsilon\right)_k(2k+\lambda+\epsilon+1)_{n-k}} $$
$$ {}_2F_0\left(\begin{matrix} k-n,-n-k-\lambda-\epsilon \\ - \end{matrix}\middle|-\frac{1}{\lambda}\right) C^{(\lambda)}_{2k+\epsilon}\left(\frac{x}{\sqrt{\lambda}}\right). \tag{3.12} $$

Using properties of the Pochhammer symbol (for $\epsilon = 0, 1$), we get:

$$ \frac{\left(\frac{1}{2}+\epsilon\right)_n (2k+\epsilon)!}{\left(\frac{1}{2}+\epsilon\right)_k} = \frac{2^{2k}k!(2n+\epsilon)!}{2^{2n}n!}. \tag{3.13} $$

Substituting (3.13) into (3.12)

$$ \begin{aligned} H_{2n+\epsilon}(x) &= \sum_{k=0}^{n} \frac{(2n+\epsilon)!(\lambda)^{n+\frac{\epsilon}{2}}}{(n-k)!(\lambda)_{2k+\epsilon}(2k+\lambda+\epsilon+1)_{n-k}} \\ &\quad {}_2F_0\left(\begin{matrix} k-n,-n-k-\lambda-\epsilon \\ - \end{matrix}\middle|-\frac{1}{\lambda}\right) C^{(\lambda)}_{2k+\epsilon}\left(\frac{x}{\sqrt{\lambda}}\right), \end{aligned} \tag{3.14} $$

where: $2n+\epsilon = N$; $2k+\epsilon = N-2r$; $2n+\epsilon = 2k+\epsilon+2r$; $n = k+r$; $-r = k-n$, the above expression can be written as:

$$ H_N(x) = \sum_{r=0}^{n} \frac{N!\lambda^{\frac{N}{2}}}{r!(\lambda)_{N-2r}(N-2r+\lambda+1)_r}\,{}_2F_0\left(\begin{matrix} -r,-N+r-\lambda \\ - \end{matrix}\middle|-\frac{1}{\lambda}\right) C^{(\lambda)}_{N-2r}\left(\frac{x}{\sqrt{\lambda}}\right) \tag{3.15} $$

If we analyze the substitutions, we see that if $\epsilon = 0$, $N = 2n$, and $n = \frac{N}{2}$. If $\epsilon = 1$, $N = 2n+1$, and $n = \frac{N-1}{2}$. from where in the sum we can replace $n$ by $[\frac{N}{2}]$ from where (3.15) we are left with:

$$ H_N(x) = \sum_{r=0}^{[\frac{N}{2}]} \frac{N!\lambda^{\frac{N}{2}}}{r!(\lambda)_{N-2r}(N-2r+\lambda+1)_r}\,{}_2F_0\left(\begin{matrix} -r,-N+r-\lambda \\ - \end{matrix}\middle|-\frac{1}{\lambda}\right) C^{(\lambda)}_{N-2r}\left(\frac{x}{\sqrt{\lambda}}\right). $$

## 3.2 Theorem (Connection formula for Gegenbauer polynomials in series of Hermite Polynomials)

$$C_N^{(\lambda)}\left(\frac{x}{\sqrt{\lambda}}\right) = \sum_{k=0}^{[\frac{N}{2}]} \frac{(-1)^r (\lambda)_{N-r}}{(r)!\lambda^{\frac{N-2r}{2}}(N-2r)!} {}_2F_0\left( \begin{array}{c} -r, N-r+\lambda \\ - \end{array} \middle| \frac{1}{\lambda}\right) H_{N-2r}(x)$$

**Proof:** Using the hypergeometric representations of Hermite and Gegenbauer polynomials for $\epsilon = 0, 1$ given is (3.7), (3.8), and using formula (2.6) with the following identification

$$[d_s] = \left\{\frac{1}{2} + \epsilon\right\}, \quad s = 1 \quad t = 0 \quad r = 0, \quad u = 0,$$

$$\omega = x^2, \quad q = 0, \quad p = 1, [a_p] = \{n + \lambda + \epsilon\}, \quad z = \frac{1}{\lambda}$$

it is obtained

$$\begin{aligned} {}_2F_1\left( \begin{array}{c} -n, n+\lambda+\epsilon \\ \frac{1}{2}+\epsilon \end{array} \middle| \frac{x^2}{\lambda}\right) &= \sum_{k=0}^{n} \binom{n}{k} \frac{(n+\lambda+\epsilon)_k}{\lambda^k} \\ &{}_2F_0\left( \begin{array}{c} k-n, k+n+\lambda+\epsilon \\ - \end{array} \middle| \frac{1}{\lambda}\right) {}_1F_1\left( \begin{array}{c} -k \\ \frac{1}{2}+\epsilon \end{array} \middle| x^2\right). \end{aligned} \tag{3.16}$$

We multiply this identity by terms that suit us to which to the left of the equality gives us (3.8), and the last part on the right of equality gives us (3.7), us gives

$$\begin{aligned} C_{2n+\epsilon}^{(\lambda)}\left(\frac{x}{\sqrt{\lambda}}\right) &= \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^{n-k} \left(\frac{1}{2}+\epsilon\right)_n (n+\lambda+\epsilon)_k (2\lambda)_{2n+\epsilon}}{(\lambda+\frac{1}{2})_n (2n+\epsilon)!\, \lambda^{\frac{\epsilon}{2}+k} 2^{2k+\epsilon} (\frac{1}{2}+\epsilon)_k} \\ &\times {}_2F_0\left( \begin{array}{c} k-n, k+n+\lambda+\epsilon \\ - \end{array} \middle| \frac{1}{\lambda}\right) H_{2k+\epsilon}(x). \end{aligned} \tag{3.17}$$

From (3.17), using properties of the Gamma function, the formula for duplicating the Pochhammer symbol, it results that:

$$C_{2n+\epsilon}^{(\lambda)}\left(\frac{x}{\sqrt{\lambda}}\right) = \sum_{k=0}^{n} \frac{(-1)^{n-k} (\lambda)_{n+k+\epsilon}}{(n-k)!\lambda^{\frac{\epsilon}{2}+k}(2k+\epsilon)!} \times {}_2F_0\left( \begin{array}{c} k-n, k+n+\lambda+\epsilon \\ - \end{array} \middle| \frac{1}{\lambda}\right) H_{2k+\epsilon}(x), \tag{3.18}$$

with the substitution

$$2n+\epsilon = N,\ 2k+\epsilon = N-2r,\ 2n+\epsilon = 2k+\epsilon+2r,\ n = k+r,\ -r = k-n$$

the above expression can be written as:

$$C_N^{(\lambda)}\left(\frac{x}{\sqrt{\lambda}}\right)=\sum_{k=0}^{n}\frac{(-1)^r(\lambda)_{N-r}}{(r)!\lambda^{\frac{N-2r}{2}}(N-2r)!}\times {}_2F_0\left(\begin{array}{c}-r,N-r+\lambda\\ -\end{array}\middle|\frac{1}{\lambda}\right)H_{N-2r}(x), \tag{3.19}$$

with a procedure similar to that of Theorem 3.1, the sum we can replace $n$ by $[\frac{N}{2}]$ from where (3.19) we are left with:

$$C_N^{(\lambda)}\left(\frac{x}{\sqrt{\lambda}}\right)=\sum_{k=0}^{[\frac{N}{2}]}\frac{(-1)^r(\lambda)_{N-r}}{(r)!\lambda^{\frac{N-2r}{2}}(N-2r)!}{}_2F_0\left(\begin{array}{c}-r,N-r+\lambda\\ -\end{array}\middle|\frac{1}{\lambda}\right)H_{N-2r}(x)$$

## 3.3 Theorem (Connection formulas between shifted Laguerre and Jacobi polynomials)

Using (2.5) with $[d_s]=\emptyset,\quad s=0,\quad [\alpha_t]=\{\gamma+1\},\quad t=1,\quad [c_r]=\emptyset,$ $r=0,\quad [\beta_u]=\emptyset,\quad u=0,\quad [a_p]=\emptyset,\quad p=0,\quad [b_q]=\{\alpha+1\},\quad q=1,$ $w=x,\quad z=1\quad \zeta=\beta+\gamma+1$, obtaining:

$$L_n^{(\alpha)}(x)=\sum_{k=0}^{n}\frac{(\alpha+1)_n(-1)^k(\beta+\gamma+1)_k}{(n-k)!(\alpha+1)_k(\beta+\gamma+1)_{2k}}$$
$$\times\,{}_2F_2\left(\begin{array}{c}k-n,k+\gamma+1\\ 2k+\beta+\gamma+2,k+\alpha+1\end{array}\middle|1\right)R_k^{(\beta,\gamma)}(x).$$

Using (2.6) with $[d_s]=\emptyset,\quad s=0,\quad [\alpha_t]=\{\gamma+1\},\quad t=1,\quad [c_r]=\emptyset,$ $r=0,\quad [\beta_u]=\emptyset,\quad u=0,\quad [a_p]=\{n+\alpha+\beta+1\},\quad p=1,\quad [b_q]=\{\beta+1\},\quad q=1,w=x,\quad z=1$, obtaining:

$$R_n^{(\alpha,\beta)}(x)=\sum_{k=0}^{n}\frac{(-1)^n(\beta+1)_n(n+\alpha+\beta+1)_k}{(n-k)!(\beta+1)_k}$$
$$\times\,{}_3F_1\left(\begin{array}{c}k-n,k+n+\alpha+\beta+1,k+\gamma+1\\ k+\beta+1\end{array}\middle|1\right)L_k^{(\gamma)}(x).$$

## 3.4 Note

Following a similar procedure as in Theorem 3.3 leads to the connection between Laguerre and Gegenbauer.

$$L_n^{(b)}(x)=\sum_{k=0}^{n}\frac{(b+1)_n(-n)_k}{(b+1)_k n!2^k(\nu)_k}{}_2F_3\left(\begin{array}{c}\triangle(2,m-n)\\ \triangle(2,b+k+1),\ \nu+k+1\end{array}\middle|\frac{1}{4}\right)G_k^{\nu}(x).$$

$$G_n^\omega(x) = \sum_{k=0}^{n} \frac{(-n)^k 2^n (a+1)_n (\omega) n}{(a+1)_k n!} {}_2F_3\left( \begin{array}{c} \triangle(2, m-n) \\ \triangle(2, -a-n),\ 1-\omega-n \end{array} \middle| \frac{1}{4} \right) L_k^a(x).$$

The notation $\triangle(r, \lambda)$, is used to abbreviate the array of $r$ parameters $\frac{(\lambda+j-1)}{r}$, $j = 1, 2, \cdot\cdot, r$.

# 4 Summary and continuity of the work

In this work, we presented the connections between the Gegenbauer polynomials and the Hermite polynomials, and formulas of connection between the Laguerre polynomials and the shifted Jacobi polynomials. In a later work, the authors hope to present the connection between the Bessel polynomials and the Hermite polynomials.

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