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Applications of Q-Ruscheweyh Symmetric Differential Operator in a Class of Starlike

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Abstract

We establish a new class of uniformly starlike functions by using a differential operator that was motivated by the authors. Moreover, we introduce a subclass with negative coefficients by fixing a second coefficient. Furthermore, we discuss the closure of this subclass under convex combinations. Finally, we investigate the extreme point values for the new class.

1 Introduction

Most recently, the researchers were motivated to study the q-calculus because it has a lot of applications in mathematics and physics. In 1908, Jackson [12], [13] presented various applications of q-calculus, as well as the q-analogues of the integral and derivative operators. Later, by applying the q-beta function, Aral and Gupta [5, 4] defined the q-Baskakov-Durrmeyer operator, and they developed the q-generalization principle of complex operators, known as the q-Gauss-Weierstrass-Picard singular integral operator. Kanas and Raducanu [14] The q-analogue of the Ruscheweyh developed and investigated the differential operator using the convolution of normalized analytic functions. Aldweby and Darus [1] investigated further the applications of the

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Ruscheweyh differential operator. In later years, several researchers have shown a significant amount of interest in this. For very recent findings concerning uniformly starlike and uniformly convex functions investigated for many new classes of functions, we refer the reader to [11], [15], [18] . Our purpose in this paper is to apply the $q-$Ruscheweyh symmetric operator introduced by Alshammari and Darus [2] and then to provide some exciting applications of this operator. Let $\mathbf{A}$ indicate the class of normalized analytic functions $\Gamma(\psi)$ in the open unit disc $\mathit{\Delta} = \{\psi : \psi \in \mathbf{C}, |\psi| < 1\}$, where $\Gamma(0) = 0$ and $\Gamma'(0) = 1$. Thus, the functions in $\mathbf{A}$ are stated by the Taylor series expansion as follows:

$$\Gamma(\psi) = \psi + \sum_{k=2}^{\infty} a_k \psi^k, \quad \psi \in \mathit{\Delta}. \tag{1.1}$$

Let $\mathbf{S} \subset \mathbf{A}$ and containing of the functions that are univalent in the open unit disc $\mathit{\Delta}$. Goodman [7, 8] defined and introduced the following subclasses of CV and ST.

Definition 1.1. *[7] A function $\Gamma(\psi)$ is said to be uniformly convex in $\mathit{\Delta}$ if $\Gamma(\psi)$ is in $\mathbb{C}V$ and has the property that for every circular arc γ contained in $\mathit{\Delta}$, with center ξ also in $\mathit{\Delta}$, the arc $f\Gamma(\gamma)$ is a convex arc with respect to $\Gamma(\xi)$.*

Definition 1.2. *[8] A function $\Gamma(\psi)$ is said to be uniformly starlike in $\mathit{\Delta}$ if $\Gamma(\psi)$ is in $\mathbb{S}T$ and has the property that for every circular arc γ contained in $\mathit{\Delta}$, with center ξ also in $\mathit{\Delta}$, the arc $\Gamma(\gamma)$ is starlike with respect to $\Gamma(\xi)$. We let $\mathbb{U}ST$ denote the class of all such functions.*

In 2018, Thomas et al. [17] proved

$$\Gamma \in \mathbb{U}ST \Longleftrightarrow \left|\frac{\psi\Gamma''(\psi)}{\Gamma'(\psi)}\right| \leq \mathbf{Re}\left\{1 + \frac{\psi\Gamma''(\psi)}{\Gamma'(\psi)}\right\}.$$

Moreover, Rϕnning [16] introduced a new class related to $\mathbb{U}CV$ stated as

$$\Gamma \in \mathcal{S}_p \Longleftrightarrow \left|\frac{\psi\Gamma'(\psi)}{\Gamma(\psi)} - 1\right| \leq \mathbf{Re}\left\{\frac{\psi\Gamma'(\psi)}{\Gamma(\psi)}\right\}.$$

Note that $\Gamma(\psi)$ is in $\mathbb{U}CV \Leftrightarrow \psi\Gamma'(\psi)$ belong to $\mathbb{S}_p$. In addition, by introducing the parameter ϖ, where $-1 \leq \varpi < 1$, Rϕnning generalized the class $\mathbb{S}_p$

$$\Gamma \in \mathcal{S}_p(\varpi) \Leftrightarrow \left|\frac{\psi\Gamma'(\psi)}{\Gamma(\psi)} - 1\right| \leq \mathbf{Re}\left\{\frac{\psi\Gamma'(\psi)}{\Gamma(\psi)} - \varpi\right\}.$$

The q-number factorial for any positive integer k is

$$[k]_q! = \begin{cases} 1 & k = 0 \\ [1]_q[2]_q[3]_q \dots [k]_q & k = 1, 2, 3, \dots \end{cases}$$

The details about quantum calculus used in this paper can be found in [6] and [10].
The Hadamard product of two functions $\Gamma(\psi)$ of the form (1) and $g(\psi)$ is of the form

$$g(\psi) = \psi + \sum_{k=2}^{\infty} b_k \psi^k, \quad \psi \in \Delta$$

as

$$(\Gamma * g)(\psi) = \psi + \sum_{k=2}^{\infty} a_k b_k \psi^k.$$

Recently, Aldweby and Darus [1] defined the q-Ruscheweyh derivative operator as follows

$$\Gamma(\psi) = \psi + \sum_{k=2}^{\infty} \frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!} a_k \psi^k.$$

Ibrahim and Darus [9] introduced the following symmetric Salagean differential operator given as

$$g(\psi) = \psi + \sum_{k=2}^{\infty} [k(\nu - (1-\nu)(-1)^k]^\mu a_k \psi^k.$$

In 2020, the authors introduced a new q derivative operator by taking the convolution between the q-Ruscheweyh derivative and symmetric Salagean differential operator as follows:

$$\Upsilon_{(q,\lambda,\nu,\mu)}\Gamma(\psi) = \psi + \sum_{k=2}^{\infty} \frac{[k+\lambda-1]_q![k(\nu - (1-\nu)(-1)^k)]^\mu}{[\lambda]_q![k-1]_q!} a_k \psi^k, \qquad (1.2)$$

where $\lambda > -1$, $0 \le \nu \le 1$, $0 < q < 1$, and $\mu = 0, 1, 2, \dots.$

Remark 1.3. *From (1.2), we can see that:*

- When $\mu = 0$, $\Upsilon_{(q,\lambda,\nu,\mu)}\Gamma(\psi)$ becomes the q-Ruscheweyh operator.

- When $\mu = 0$ and $q \to 1$, $\Upsilon_{(q,\lambda,\nu,\mu)}\Gamma(\psi)$ becomes the Ruscheweyh operator.
- When $\mu = 0$ and $\lambda = 1$ $\Upsilon_{(q,\lambda,\nu,\mu)}\Gamma(\psi)$ becomes the q-Salagean operator.
- When $\mu = 0$, $\lambda = 1$, and $q \to 1$, $\Upsilon_{(q,\lambda,\nu,\mu)}\Gamma(\psi)$ becomes the Salagean operator.

This operator has recently been used to study the second Hankel determinant by the authors [3].
In this paper, we will use the differential operator that was introduced by Al Shammari and Darus [2] in our main results.

Definition 1.4. *Let $S(q, \lambda, \nu, \mu, \varpi, \epsilon)$ indicate the subclass of S containing functions $\Gamma(\psi)$ of the form (1.1) and satisfying*

$$\Re\left\{\frac{\psi\partial_q(\Upsilon_{(q,\lambda,\nu,\mu)}\Gamma(\psi))}{\Upsilon_{(q,\lambda,\nu,\mu)}\Gamma(\psi)} - \varpi\right\} > \epsilon\left|\frac{\psi\partial_q(\Upsilon_{(q,\lambda,\nu,\mu)}\Gamma(\psi))}{\Upsilon_{(q,\lambda,\nu,\mu)}\Gamma(\psi)} - 1\right|.$$

Moreover, we assume $TS^*(q, \lambda, \nu, \mu, \varpi, \epsilon) = S(q, \lambda, \nu, \mu, \varpi, \epsilon) \cap T$, where T represents the subclass of S containing functions of the form

$$\Gamma(\psi) = \psi - \sum_{k=2}^{\infty} a_k \psi^k, \quad a_k \geq 0, \quad \forall \;\; k \geq 2. \tag{1.3}$$

Our main objective in this paper is to obtain the necessary and sufficient conditions for the functions $\Gamma(\psi) \in TS^*(q, \lambda, \nu, \mu, \varpi, \epsilon)$. Moreover, by fixing the second coefficient, we get the extreme points for the function $\Gamma(\psi) \in TS^*(q, \lambda, \nu, \mu, \varpi, \epsilon)$.

2 Main results

2.1 The Class $S(q, \lambda, \nu, \mu, \varpi, \epsilon)$

In this part of the article, we will get a condition that is both necessary and sufficient for the function $\Gamma(\psi)$ in the classes $TS^*(q, \lambda, \nu, \mu, \varpi, \epsilon)$.

Theorem 2.1. *Let $\Gamma(\psi)$ be a function of the form (1.1) belonging to $S(q, \lambda, \nu, \mu, \varpi, \epsilon)$. If is true only if*

$$\sum_{k=2}^{\infty} [[k]_q(1+\epsilon) - (\varpi + \epsilon)] \frac{[k+\lambda-1]_q![k(\nu - (1-\nu)(-1)^k]^{\mu}}{[\lambda]_q![k-1]_q!}|a_k| \leq 1 - \varpi, \tag{2.4}$$

where $-1 < \varpi \leq 1$ and $\epsilon \geq 0$.

Proof.
To reach our result, we only need to show that

$$\epsilon\left|\frac{\psi\partial_q(\Upsilon_{(q,\lambda,\nu,\mu)}\Gamma(\psi))}{\Upsilon_{(q,\lambda,\nu,\mu)}\Gamma(\psi)}-1\right|-\Re\left\{\frac{\psi\partial_q(\Upsilon_{(q,\lambda,\nu,\mu)}\Gamma(\psi))}{\Upsilon_{(q,\lambda,\nu,\mu)}\Gamma(\psi)}-1\right\}\leq 1-\varpi.$$

Starting from the left hand side, we get

$$\begin{aligned}&\epsilon\left|\frac{\psi\partial_q(\Upsilon_{(q,\lambda,\nu,\mu)}\Gamma(\psi))}{\Upsilon_{(q,\lambda,\nu,\mu)}\Gamma(\psi)}-1\right|-\Re\left\{\frac{\psi\partial_q(\Upsilon_{(q,\lambda,\nu,\mu)}\Gamma(\psi))}{\Upsilon_{(q,\lambda,\nu,\mu)}\Gamma(\psi)}-1\right\}\\ &\leq\epsilon\left|\frac{\psi\partial_q(\Upsilon_{(q,\lambda,\nu,\mu)}\Gamma(\psi))}{\Upsilon_{(q,\lambda,\nu,\mu)}\Gamma(\psi)}-1\right|+\left|\frac{\psi\partial_q(\Upsilon_{(q,\lambda,\nu,\mu)}\Gamma(\psi))}{\Upsilon_{(q,\lambda,\nu,\mu)}\Gamma(\psi)}-1\right|.\end{aligned}$$

Thus

$$\begin{aligned}\epsilon\left|\frac{\psi\partial_q(\Upsilon_{(q,\lambda,\nu,\mu)}\Gamma(\psi))}{\Upsilon_{(q,\lambda,\nu,\mu)}\Gamma(\psi)}-1\right|-\Re\left\{\frac{\psi\partial_q(\Upsilon_{(q,\lambda,\nu,\mu)}\Gamma(\psi))}{\Upsilon_{(q,\lambda,\nu,\mu)}\Gamma(\psi)}-1\right\}&\leq(1+\epsilon)\left|\frac{\psi\partial_q(\Upsilon_{(q,\lambda,\nu,\mu)}\Gamma(\psi))}{\Upsilon_{(q,\lambda,\nu,\mu)}\Gamma(\psi)}-1\right|\\ &\leq(1+\epsilon)\frac{\sum_{k=2}^{\infty}([k]_q-1)\frac{[k+\lambda-1]_q![k(\nu-(1-\nu)(-1)^k]^{\mu}}{[\lambda]_q![k-1]_q!}|a_k|}{1-\sum_{k=2}^{\infty}\frac{[k+\lambda-1]_q![k(\nu-(1-\nu)(-1)^k]^{\mu}}{[\lambda]_q![k-1]_q!}|a_k|},\end{aligned}$$

and

$$\begin{aligned}&\epsilon\left|\frac{\psi\partial_q(\Upsilon_{(q,\lambda,\nu,\mu)}\Gamma(\psi))}{\Upsilon_{(q,\lambda,\nu,\mu)}\Gamma(\psi)}-1\right|-\Re\left\{\frac{\psi\partial_q(\Upsilon_{(q,\lambda,\nu,\mu)}\Gamma(\psi))}{\Upsilon_{(q,\lambda,\nu,\mu)}\Gamma(\psi)}-1\right\}\\ &\leq\frac{\sum_{k=2}^{\infty}[[k]_q(1+\epsilon)-(1+\epsilon)]\frac{[k+\lambda-1]_q![k(\nu-(1-\nu)(-1)^k]^{\mu}}{[\lambda]_q![k-1]_q!}|a_k|}{1-\sum_{k=2}^{\infty}\frac{[k+\lambda-1]_q![k(\nu-(1-\nu)(-1)^k]^{\mu}}{[\lambda]_q![k-1]_q!}|a_k|}.\end{aligned}$$

The last formula is bounded above by $(1-\varpi)$, if

$$\sum_{k=2}^{\infty}[[k]_q(1+\epsilon)-(\varpi+\epsilon)]\frac{[k+\lambda-1]_q![k(\nu-(1-\nu)(-1)^k]^{\mu}}{[\lambda]_q![k-1]_q!}|a_k|\leq 1-\varpi,$$

and the proof is complete.

Theorem 2.2. *The function $\Gamma(\psi)$ is said to be a subset of class $TS^*(q,\lambda,\nu,\mu,\varpi,\epsilon)$, if it satisfies the necessary and sufficient conditions as follows:*

$$\sum_{k=2}^{\infty}[[k]_q(1+\epsilon)-(\varpi+\epsilon)]\frac{[k+\lambda-1]_q![k(\nu-(1-\nu)(-1)^k]^{\mu}}{[\lambda]_q![k-1]_q!}|a_k|\leq 1-\varpi, \tag{2.5}$$

where $-1<\varpi\leq 1$ and $\epsilon\geq 0$.

Proof.
In Theorem 2.1, we simply have to show the necessary condition. Without loss of generality, let $\Gamma \in TS^*(q,\lambda,\nu,\mu,\varpi,\epsilon)$ and let ψ be a real number. Then

$$\Re\{\frac{\psi\partial_q(\Upsilon_{(q,\lambda,\nu,\mu)}\Gamma(\psi))}{\Upsilon_{(q,\lambda,\nu,\mu)}\Gamma(\psi)} - \varpi\} > \epsilon\left|\frac{\psi\partial_q(\Upsilon_{(q,\lambda,\nu,\mu)}\Gamma(\psi))}{\Upsilon_{(q,\lambda,\nu,\mu)}\Gamma(\psi)} - 1\right|$$

$$\frac{\psi - \sum_{k=2}^{\infty}[k]_q\frac{[k+\lambda-1]_q![k(\nu-(1-\nu)(-1)^k]^\mu}{[\lambda]_q![k-1]_q!}a_k\psi^k}{\psi - \sum_{k=2}^{\infty}\frac{[k+\lambda-1]_q![k(\nu-(1-\nu)(-1)^k]^\mu}{[\lambda]_q![k-1]_q!}a_k\psi^k} - \varpi$$
$$> \epsilon\left|\frac{\psi - \sum_{k=2}^{\infty}[k]_q\frac{[k+\lambda-1]_q![k(\nu-(1-\nu)(-1)^k]^\mu}{[\lambda]_q![k-1]_q!}a_k\psi^k}{\psi - \sum_{k=2}^{\infty}\frac{[k+\lambda-1]_q![k(\nu-(1-\nu)(-1)^k]^\mu}{[\lambda]_q![k-1]_q!}a_k\psi^k} - 1\right|.$$

Letting $\psi \to 1$, we obtain the required inequality

$$\sum_{k=2}^{\infty}[[k]_q(1+\epsilon) - (\varpi+\epsilon)]\frac{[k+\lambda-1]_q![k(\nu-(1-\nu)(-1)^k]^\mu}{[\lambda]_q![k-1]_q!}|a_k| \leq 1-\varpi.$$

Corollary 2.3. *We assume that the function $\Gamma(\psi)$ be of the form (1.3) $\in TS^*(q,\lambda,\nu,\mu,\varpi,\epsilon)$. Then*

$$a_k \leq \frac{(1-\varpi)[\lambda]_q![k-1]_q!}{[[k]_q(1+\epsilon)-(\varpi+\epsilon)][k+\lambda-1]_q![k(\nu-(1-\nu)(-1)^k]^\mu}, \qquad k \geq 2.$$

Corollary 2.4.

$$a_2 \leq \frac{(1-\varpi)}{[[2]_q(1+\epsilon)-(\varpi+\epsilon)]\,[\lambda+1]_q[4\nu-2]^\mu}.$$

2.2 The Class $TS_d^*(q,\lambda,\nu,\mu,\varpi,\epsilon)$

By setting the second coefficient $\in TS^*(q,\lambda,\nu,\mu,\varpi,\epsilon)$, we establish a new subclass, called $TS_d^*(q,\lambda,\nu,\mu,\varpi,\epsilon)$, as follows:

Definition 2.5. *Let $0 < d \leq 1$ including $\Gamma(\psi) \in TS^*(q,\lambda,\nu,\mu,\varpi,\epsilon)$. Then $\Gamma(\psi) \in TS_d^*(q,\lambda,\nu,\mu,\varpi,\epsilon)$, which has the following form*

$$\Gamma(\psi) = \psi - \frac{d(1-\varpi)}{[[2]_q(1+\epsilon)-(\varpi+\epsilon)]\,[\lambda+1]_q[4\nu-2]^\mu}\psi^2 - \sum_{k=3}^{\infty}a_k\psi^k. \quad (2.6)$$

Theorem 2.6. *Let the function* $\Gamma(\psi)$ *be established by (2.6). Subsequently,* $\Gamma(\psi) \in TS_d^*(q,\lambda,\nu,\mu,\varpi,\epsilon)$ *if and only if*

$$\sum_{k=3}^{\infty} [[k]_q(1+\epsilon) - (\varpi+\epsilon)] \frac{[k+\lambda-1]_q![k(\nu-(1-\nu)(-1)^k]^{\mu}}{[\lambda]_q![k-1]_q!} a_k \le (1-d)(1-\varpi). \tag{2.7}$$

Proof. Substituting

$$a_2 = \frac{d(1-\varpi)}{[[2]_q(1+\epsilon) - (\varpi+\epsilon)]\,[\lambda+1]_q[4\nu-2]^{\mu}}$$

in (2.5) with a simple calculation will give the result. □

Corollary 2.7. *Let the function* $\Gamma(\psi)$ *be defined by (2.6) is in* $TS_d^*(q,\lambda,\nu,\mu,\varpi,\epsilon)$. *Then*

$$a_k \le \frac{(1-d)(1-\varpi)[\lambda]_q![k-1]_q!}{[[k]_q(1+\epsilon) - (\varpi+\epsilon)][k+\lambda-1]_q![k(\nu-(1-\nu)(-1)^k]^{\mu}}, \qquad k \ge 3. \tag{2.8}$$

Theorem 2.8. *The class* $TS_d^*(q,\lambda,\nu,\mu,\varpi,\epsilon)$ *is closed under a convex linear combination.*

Proof.
Assume the functions $\Gamma(\psi)$ and $g(\psi) \in TS_d^*(q,\lambda,\nu,\mu,\varpi,\epsilon)$. Let $\Gamma(\psi)$ be stated by (2.6) and

$$g(\psi) = \psi - \frac{d(1-\varpi)}{[[2]_q(1+\epsilon) - (\varpi+\epsilon)]\,[\lambda+1]_q[4\nu-2]^{\mu}}\psi^2 - \sum_{k=3}^{\infty} \rho_k \psi^k, \tag{2.9}$$

where $\rho_k \ge 0$. It suffices to show that, for $0 \le \omega \le 1$, the function

$$I(\psi) = \omega\Gamma(\psi) + (1-\omega)g(\psi), \tag{2.10}$$

also belongs to $TS_d^*(q,\lambda,\nu,\mu,\varpi,\epsilon)$. From (2.6), (2.9) and (2.10), we have

$$I(\psi) = \psi - \frac{d(1-\varpi)}{[[2]_q(1+\epsilon) - (\varpi+\epsilon)]\,[\lambda+1]_q[4\nu-2]^{\mu}}\psi^2 - \sum_{k=3}^{\infty}\{\omega a_k + (1-\omega)\rho_k\}\psi^k. \tag{2.11}$$

Since $\Gamma(\psi)$ and $g(\psi) \in TS_d^*(q,\lambda,\nu,\mu,\varpi,\epsilon)$ and $0 \le \omega \le 1$, by applying Theorem 2.6, we get

$$\sum_{k=3}^{\infty}[[k]_q(1+\epsilon)-(\varpi+\epsilon)]\frac{[k+\lambda-1]_q![k(\nu-(1-\nu)(-1)^k]^\mu}{[\lambda]_q![k-1]_q!}\{\omega a_k+(1-\omega)\rho_k\} \le (1-d)(1-\varpi). \tag{2.12}$$

Once more, based on Theorem 2.6 and (2.12), we obtain $I(\psi) \in TS_d^*(q,\lambda,\nu,\mu,\varpi,\epsilon)$.

Theorem 2.9. *Assume the function*

$$\Gamma_r(\psi) = \psi - \frac{d(1-\varpi)}{[[2]_q(1+\epsilon)-(\varpi+\epsilon)]\,[\lambda+1]_q[4\nu-2]^\mu}\psi^2 - \sum_{k=3}^{\infty} a_{k,r}\psi^k, \quad a_{k,r} \ge 0 \tag{2.13}$$

be in the class $TS_d^*(q,\lambda,\nu,\mu,\varpi,\epsilon)$, for every ($r = 1,2,3,...,s$). Thus the function $G(\psi)$ established by

$$G(\psi) = \sum_{r=1}^{s}\vartheta_r\Gamma_r(\psi), \tag{2.14}$$

is also $\in TS_d^*(q,\lambda,\nu,\mu,\varpi,\epsilon)$, where

$$\sum_{r=1}^{s}\vartheta_r = 1. \tag{2.15}$$

Proof. From (2.13), (2.14) and (2.15), we get

$$F(\psi) = \psi - \frac{d(1-\varpi)}{[[2]_q(1+\epsilon)-(\varpi+\epsilon)]\,[\lambda+1]_q[4\nu-2]^\mu}\psi^2 - \sum_{k=3}^{\infty}\left(\sum_{r=1}^{s}\vartheta_r a_{k,r}\right)\psi^k.$$

Since $\Gamma_r(\psi) \in TS_d^*(q,\lambda,\nu,\mu,\varpi,\epsilon)$ for every $r = 1,2,3,...,s$ Theorem 2.6 denotes

$$\sum_{k=3}^{\infty}[[k]_q(1+\epsilon)-(\varpi+\epsilon)]\frac{[k+\lambda-1]_q![k(\nu-(1-\nu)(-1)^k]^\mu}{[\lambda]_q![k-1]_q!}a_{k,r} \le (1-d)(1-\varpi). \tag{2.16}$$

Now, we show that $G(\psi)$ fulfills the condition of (2.8) that will lead us to $F(\psi) \in TS_d^*(q,\lambda,\nu,\mu,\varpi,\epsilon)$

$$\sum_{k=3}^{\infty}[[k]_q(1+\epsilon)-(\varpi+\epsilon)]\frac{[k+\lambda-1]_q![k(\nu-(1-\nu)(-1)^k]^\mu}{[\lambda]_q![k-1]_q!}\left(\sum_{r=1}^{s}\vartheta_r a_{k,r}\right)$$

$$= \sum_{r=1}^{s} \vartheta_r \left(\sum_{k=3}^{\infty} [[k]_q(1+\epsilon) - (\varpi+\epsilon)] \frac{[k+\lambda-1]_q![k(\nu-(1-\nu)(-1)^k]^{\mu}}{[\lambda]_q![k-1]_q!} a_{k,r} \right). \tag{2.17}$$

Using (2.15), (2.16) in (2.17), we obtain

$$\sum_{k=3}^{\infty} [[k]_q(1+\epsilon) - (\varpi+\epsilon)] \frac{[k+\lambda-1]_q![k(\nu-(1-\nu)(-1)^k]^{\mu}}{[\lambda]_q![k-1]_q!} \left(\sum_{r=1}^{s} \vartheta_r a_{k,r} \right) \leq (1-d)(1-\varpi).$$

Thus $G(\psi) \in TS_d^*(q,\lambda,\nu,\mu,\varpi,\epsilon)$. □

Theorem 2.10. *Let*

$$\Gamma_2(\psi) = \psi - \frac{d(1-\varpi)}{[[2]_q(1+\epsilon) - (\varpi+\epsilon)]\,[\lambda+1]_q[4\nu-2]^{\mu}} \psi^2 \tag{2.18}$$

and $\Gamma_k(\psi)$ *is*

$$= \psi - \frac{d(1-\varpi)}{[[2]_q(1+\epsilon) - (\varpi+\epsilon)]\,[\lambda+1]_q[4\nu-2]^{\mu}} \psi^2 \tag{2.19}$$

$$- \frac{(1-d)(1-\varpi)[\lambda]_q![k-1]_q!}{[[k]_q(1+\epsilon) - (\varpi+\epsilon)][k+\lambda-1]_q![k(\nu-(1-\nu)(-1)^k]^{\mu}} \psi^k, \tag{2.20}$$

for $k = 3, 4, \ldots$ *. Then* $\Gamma(\psi) \in TS_d^*(q,\lambda,\nu,\mu,\varpi,\epsilon) \Leftrightarrow$ *can be written as*

$$\Gamma(\psi) = \sum_{k=2}^{\infty} \vartheta_k f_k(\psi), \tag{2.21}$$

where, $\vartheta_k \geq 0$ *and* $\sum_{k=2}^{\infty} \vartheta_k = 1$.

Proof.
Suppose that $\Gamma(\psi)$ is in the form (2.21). Substituting (2.18) and (2.19) in (2.21), we have

$$\Gamma(\psi) = \psi - \sum_{k=2}^{\infty} A_k \psi^k, \tag{2.22}$$

where

$$A_2 = \frac{d(1-\varpi)}{[[2]_q(1+\epsilon) - (\varpi+\epsilon)]\,[\lambda+1]_q[4\nu-2]^{\mu}} \tag{2.23}$$

and

$$A_k = \frac{(1-d)(1-\varpi)[\lambda]_q![k-1]_q!}{[[k]_q(1+\epsilon) - (\varpi+\epsilon)][k+\lambda-1]_q![k(\nu-(1-\nu)(-1)^k]^{\mu}} \quad k \geq 3. \tag{2.24}$$

In order to establish that $\Gamma(\psi)$ is in the class $TS^*_d(q,\lambda,\nu,\mu,\varpi,\epsilon)$, it is enough to show that it satisfies the condition of Theorem 2.10. Consider

$$\sum_{k=2}^{\infty}[[k]_q(1+\epsilon)-(\varpi+\epsilon)]\frac{[k+\lambda-1]_q![k(\nu-(1-\nu)(-1)^k]^\mu}{[\lambda]_q![k-1]_q!}A_k$$
$$=d(1-\varpi)+\sum_{k=3}^{\infty}\vartheta_k(1-d)(1-\varpi).$$

Since $\sum_{k=2}^{\infty}\vartheta_k=1$, we can write the above equation as

$$\sum_{k=2}^{\infty}[[k]_q(1+\epsilon)-(\varpi+\epsilon)]\frac{[k+\lambda-1]_q![k(\nu-(1-\nu)(-1)^k]^\mu}{[\lambda]_q![k-1]_q!}A_k$$
$$=(1-\varpi)[d+(1-\vartheta_2)(1-d)]\leq(1-\varpi).$$

Thus $\Gamma(\psi)\in TS^*_d(q,\lambda,\nu,\mu,\varpi,\epsilon)$. On the other hand, we assume that $\Gamma(\psi)$, defined by (2.6), belong to the class $TS^*_d(q,\lambda,\nu,\mu,\varpi,\epsilon)$. Then, by using (2.8), we get

$$a_k\leq\frac{(1-d)(1-\varpi)[\lambda]_q![k-1]_q!}{[[k]_q(1+\epsilon)-(\varpi+\epsilon)][k+\lambda-1]_q![k(\nu-(1-\nu)(-1)^k]^\mu},\qquad k\geq 3. \tag{2.25}$$

By taking

$$\vartheta_k=\frac{[[k]_q(1+\epsilon)-(\varpi+\epsilon)][k+\lambda-1]_q![k(\nu-(1-\nu)(-1)^k]^\mu a_k}{(1-d)(1-\varpi)[\lambda]_q![k-1]_q!}, \tag{2.26}$$

and

$$\vartheta_2=1-\sum_{k=3}^{\infty}\vartheta_k, \tag{2.27}$$

we have (2.21). The proof is complete.

Corollary 2.11. *According to Theorem 2.10, functions $\Gamma_k(\psi)$, where $k\geq 2$ are the extreme points of the class $TS^*_d(q,\lambda,\nu,\mu,\varpi,\epsilon)$.*

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