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Integrals Associated with Complex Multivariate Beta Function

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Abstract

In this article, we evaluate integrals which are connected to the complex multivariate beta integral. These results are useful in computing expected values of trace functions of complex random matrices.

1 Introduction

The complex multivariate generalization of the beta function is defined by

$$\begin{aligned}\widetilde{B}_m(a,b) &= \int_{0<X^{\mathsf{H}}=X<I_m} \det(X)^{a-m}\det(I_m-X)^{b-m}\,\mathrm{d}X\\ &= \frac{\widetilde{\Gamma}_m(a)\widetilde{\Gamma}_m(b)}{\widetilde{\Gamma}_m(a+b)} = \widetilde{B}_m(b,a),\end{aligned}$$

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where $\det(A)$ = determinant of A; A^{H} denotes conjugate transpose of A, $A = A^{\mathsf{H}} > 0$ means that A is Hermitian positive definite, $\mathrm{Re}(a) > m-1$ and $\mathrm{Re}(b) > m-1$. The complex multivariate gamma function $\widetilde{\Gamma}_m(a)$ used above is defined by $\widetilde{\Gamma}_m(a) = \pi^{m(m-1)/2}\prod_{i=1}^m \Gamma(a-i+1)$, $\mathrm{Re}(a) > m-1$. By substituting $X = (I_m+Y)^{-1}Y$ with the Jacobian $J(X \to Y) = \det(I_m + Y)^{-2m}$, one can also express the complex multivariate beta function as

$$\widetilde{B}_m(a,b) = \int_{Y^{\mathsf{H}}=Y>0} \det(Y)^{a-m}\det(I_m+Y)^{-a-b}\,\mathrm{d}Y.$$

Several properties and results on complex multivariate beta function are given in [11] and [12]. The complex multivariate beta function frequently occurs in multivariate analysis in the density of complex random matrices (for example, see [3]).

In this article, we evaluate integrals which are closely connected to complex multivariate beta integral. In multivariate statistical analysis these integrals are useful in computing expected values of scalar valued functions of random matrices. For an $m\times m$ Hermitian matrix G, define $\widetilde{f}_1$, $\widetilde{f}_2$ and $\widetilde{f}_3$ as

$$\widetilde{f}_1(g,h,i,j) = \int_{0<X^{\mathsf{H}}=X<I_m} \frac{\det(X)^{a-m}\det(I_m-X)^{b-m}}{\widetilde{B}_m(a,b)}(\mathrm{tr}(GX)^h)^g(\mathrm{tr}(GX)^j)^i\mathrm{d}X$$

$$\widetilde{f}_2(g,h,i,j) = \int_{0<X^{\mathsf{H}}=X<I_m} \frac{\det(X)^{a-m}\det(I_m-X)^{b-m}}{\widetilde{B}_m(a,b)}(\mathrm{tr}(GX^{-1})^h)^g(\mathrm{tr}(GX^{-1})^j)^i\mathrm{d}X,$$

$$\widetilde{f}_3(g,h,i,j) = \int_{X^{\mathsf{H}}=X>0} \frac{\det(X)^{a-m}\det(I_m+X)^{-a-b}}{\widetilde{B}_m(a,b)}(\mathrm{tr}(GX)^h)^g(\mathrm{tr}(GX)^j)^i\mathrm{d}X,$$

respectively, where g,h,i,j are non-negative integers, and a and b are complex numbers with suitable restrictions.

In Sections 3, we give explicit expressions for $\widetilde{f}_1$, $\widetilde{f}_2$, and $\widetilde{f}_3$ for specific values of g,h,i and j. These results can be used to evaluate expected values of functions of complex beta matrices, as illustrated in Section 4.

2 Some known results

For an ordered partition κ of k, $\kappa = (k_1,\ldots,k_m)$, $k_1 \geq \cdots \geq k_m \geq 0$, $k_1+\cdots+k_m = k$, $\widetilde{\Gamma}_m(a,\kappa)$ and $\widetilde{\Gamma}_m(a,-\kappa)$ are defined by $\widetilde{\Gamma}_m(a,\kappa) = [a]_\kappa\widetilde{\Gamma}_m(a)$, $\widetilde{\Gamma}_m(a,0) = \widetilde{\Gamma}_m(a)$ and $\widetilde{\Gamma}_m(a,-\kappa) = \frac{(-1)^k\,\widetilde{\Gamma}_m(a)}{[-a+m]_\kappa}$, $\mathrm{Re}(a) > k_1+m-1$.

The *complex generalized hypergeometric coefficient* $[a]_\kappa$ is defined as

$$[a]_\kappa = \prod_{i=1}^{\ell(\kappa)} (a - i + 1)_{k_i}, \tag{2.1}$$

where $\ell(\kappa)$ is the number of non-zero k_j's and $(a)_r = a(a+1)\cdots(a+r-1) = (a)_{r-1}(a+r-1)$ for $r = 1, 2, \ldots$, and $(a)_0 = 1$. Using (2.1), the computation of $[a]_\kappa$ and $[-a+m]_\kappa$ can be done for ordered partitions of k. For example, for $k = 2$ and $k = 3$, $[a]_{(2)} = a(a+1)$, $[a]_{(1^2)} = a(a-1)$, $[a]_{(3)} = a(a+1)(a+2)$, $[a]_{(2,1)} = a(a+1)(a-1)$ and $[a]_{(1^3)} = a(a-1)(a-2)$.

For a Hermitian matrix T of order m, $\mathrm{Re}(a) > m-1$, $\mathrm{Re}(b) > m-1$, $\mathrm{Re}(c) > k_1 + m - 1$, we have ([5],[7]):

$$\int_{0<R^{\mathsf{H}}=R<I_m} \det(R)^{a-m} \det(I_m - R)^{b-m} \widetilde{C}_\kappa(TR)\, \mathrm{d}R = \frac{\widetilde{\Gamma}_m(a,\kappa)\widetilde{\Gamma}_m(b)}{\widetilde{\Gamma}_m(a+b,\kappa)} \widetilde{C}_\kappa(T), \tag{2.2}$$

$$\begin{aligned}\int_{0<R^{\mathsf{H}}=R<I_m} &\det(R)^{c-m} \det(I_m - R)^{b-m} \widetilde{C}_\kappa(R^{-1}T)\, \mathrm{d}R \\ &= \frac{\widetilde{\Gamma}_m(c,-\kappa)\widetilde{\Gamma}_m(b)}{\widetilde{\Gamma}_m(c+b,-\kappa)} \widetilde{C}_\kappa(T) = \frac{\widetilde{\Gamma}_m(c)\widetilde{\Gamma}_m(b)[-c-b+m]_\kappa}{\widetilde{\Gamma}_m(c+b)[-c+m]_\kappa} \widetilde{C}_\kappa(T),\end{aligned} \tag{2.3}$$

$$\begin{aligned}\int_{R^{\mathsf{H}}=R>0} &\det(R)^{a-m} \det(I_m + R)^{-a-c} \widetilde{C}_\kappa(RT)\, \mathrm{d}R \\ &= \frac{\widetilde{\Gamma}_m(c,-\kappa)\widetilde{\Gamma}_m(a,\kappa)}{\widetilde{\Gamma}_m(c)\widetilde{\Gamma}_m(a)} \widetilde{C}_\kappa(T) = \frac{(-1)^k [a]_\kappa}{[-c+m]_\kappa} \widetilde{C}_\kappa(T),\end{aligned} \tag{2.4}$$

where $\widetilde{C}_\kappa(X)$ is the zonal polynomial ([5]) of an $m \times m$ complex symmetric matrix X corresponding to the ordered partition $\kappa = (k_1, \ldots, k_m)$, $k_1 + \cdots + k_m = k$, $k_1 \geq \cdots \geq k_m \geq 0$. For small values of k, explicit formulas for $\widetilde{C}_\kappa(X)$ are available in [8] as

$$\begin{aligned}
\widetilde{C}_{(1)}(X) &= \mathrm{tr}(X), \\
\widetilde{C}_{(2)}(X) &= \frac{1}{2}[(\mathrm{tr}(X))^2 + \mathrm{tr}(X^2)], \\
\widetilde{C}_{(1^2)}(X) &= \frac{1}{2}[(\mathrm{tr}(X))^2 - \mathrm{tr}(X^2)], \\
\widetilde{C}_{(3)}(X) &= \frac{1}{6}[(\mathrm{tr}(X))^3 + 3\,\mathrm{tr}(X)\,\mathrm{tr}(X^2) + 2\,\mathrm{tr}(X^3)], \\
\widetilde{C}_{(2,1)}(X) &= \frac{2}{3}[(\mathrm{tr}(X))^3 - \mathrm{tr}(X^3)], \\
\widetilde{C}_{(1^3)}(X) &= \frac{1}{6}[(\mathrm{tr}(X))^3 - 3\,\mathrm{tr}(X)\,\mathrm{tr}(X^2) + 2\,\mathrm{tr}(X^3)],
\end{aligned}$$

By using the above expressions for zonal polynomials, we can express $\operatorname{tr}(X^2)$, $\operatorname{tr}(X^3)$ and $\operatorname{tr}(X)\operatorname{tr}(X^2)$ as

$$\begin{aligned}\operatorname{tr}(X^2) &= \widetilde{C}_{(2)}(X) - \widetilde{C}_{(1^2)}(X),\\ \operatorname{tr}(X^3) &= \widetilde{C}_{(3)}(X) - \frac{1}{2}\widetilde{C}_{(2,1)}(X) + \widetilde{C}_{(1^3)}(X),\\ \operatorname{tr}(X)\operatorname{tr}(X^2) &= \widetilde{C}_{(3)}(X) - \widetilde{C}_{(1^3)}(X).\end{aligned}$$

Similarly, $(\operatorname{tr}(X))^2\operatorname{tr}(X^2)$, $(\operatorname{tr}(X^2))^2$, $\operatorname{tr}(X)\operatorname{tr}(X^3)$, $\operatorname{tr}(X^4)$ can be expressed in terms of zonal polynomials of order 4.

3 Evaluation of $\widetilde{f}_i(g, h, i, j)$, $i = 1, 2, 3$

Consider the integral

$$\begin{aligned}\widetilde{f}_1(1, 2, 0, -) &= \int_{0<X^{\mathsf{H}}=X<I_m} \frac{\det(X)^{a-m}\det(I_m - X)^{b-m}}{\widetilde{B}_m(a, b)}\operatorname{tr}((GX)^2)\, \mathrm{d}X\\ &= \frac{[a]_{(2)}}{[a+b]_{(2)}}\widetilde{C}_{(2)}(G) - \frac{[a]_{(1^2)}}{[a+b]_{(1^2)}}\widetilde{C}_{(1^2)}(G).\end{aligned}$$

where the second line has been obtained by writing $\operatorname{tr}((GX)^2)$ in terms of zonal polynomials and integrating resulting expression by using (2.2). Now, substituting for $[a]_{(2)}$, $[a+b]_{(2)}$, $[a]_{(1^2)}$, $[a+b]_{(1^2)}$, $\widetilde{C}_{(2)}(G)$ and $\widetilde{C}_{(1^2)}(G)$ above, we have

$$\widetilde{f}_1(1, 2, 0, -) = \frac{a[b(\operatorname{tr}(G))^2 + [a(a+b) - 1]\operatorname{tr}(G^2)]}{(a+b-1)(a+b)(a+b+1)}.$$

Similarly, we obtain

$$\begin{aligned}\widetilde{f}_1(1, 3, 0, -) = \frac{a}{(a+b-2)_5}&[b(b-a)(\operatorname{tr}(G))^3 + 3b[a(a+b) - 2]\operatorname{tr}(G)\operatorname{tr}(G^2)\\ &+ \{[a(a+b) - 2]^2 + b^2 - a^2\}\operatorname{tr}(G^3)],\end{aligned}$$

$$\begin{aligned}\widetilde{f}_1(1, 2, 1, 1) = \frac{a}{(a+b-2)_5}&[b\{a(a+b) - 2\}\{(\operatorname{tr}(G))^3 + 2\operatorname{tr}(G^3)\}\\ &+ [(a^2+2)\{(a+b)^2 + 2\} - 9a(a+b)]\operatorname{tr}(G)\operatorname{tr}(G^2)].\end{aligned}$$

Writing $\operatorname{tr}((X^{-1}G)^2)$ in terms of zonal polynomials and integrating the resulting expression by using (2.3), we obtain

$$\begin{aligned}\widetilde{f}_2(1, 2, 0, -) &= \int_{0<X^{\mathsf{H}}=X<I_m} \frac{\det(X)^{a-m}\det(I_m - X)^{b-m}}{\widetilde{B}_m(a, b)}\operatorname{tr}((X^{-1}G)^2)\, \mathrm{d}X\\ &= \frac{[-a-b+m]_{(2)}}{[-a+m]_{(2)}}\widetilde{C}_{(2)}(G) - \frac{[-a-b+m]_{(1^2)}}{[-a+m]_{(1^2)}}\widetilde{C}_{(1^2)}(G).\end{aligned}$$

Now, substituting for $[-a+m]_{(2)}$, $[-a-b+m]_{(2)}$, $[-a+m]_{(1^2)}$, $[-a-b+m]_{(1^2)}$, $\widetilde{C}_{(2)}(G)$ and $\widetilde{C}_{(1^2)}(G)$ above, we have

$$\widetilde{f}_2(1,2,0,-) = \frac{(a+b-m)[b(\operatorname{tr}(G))^2 + [(a-m)(a+b-m)-1]\operatorname{tr}(G^2)]}{(a-m)[(a-m)^2-1]}.$$

Similarly, we obtain

$$\begin{aligned}\widetilde{f}_2(1,3,0,-) = \frac{a+b-m}{(a-m-2)_5}&[b(a+2b-m)\{(\operatorname{tr}(G))^3 + 2\operatorname{tr}(G^3)\}\\ &+ 3b\{(a-m)(a+b-m)-2\}\operatorname{tr}(G)\operatorname{tr}(G^2)\\ &+ \{(a-m)^2-4\}\{(a+b-m)^2-1\}\operatorname{tr}(G^3)],\end{aligned}$$

$$\begin{aligned}\widetilde{f}_2(1,2,1,1) = \frac{(a+b-m)}{(a-m-2)_5}&[b\{(a-m)(a+b-m)-2\}\{(\operatorname{tr}(G))^3 + 2\operatorname{tr}(G^3)\}\\ &+ [\{(a-m)^2+2\}\{(a+b-m)^2+2\}\\ &- 9(a-m)(a+b-m)]\operatorname{tr}(G)\operatorname{tr}(G^2)],\end{aligned}$$

$$\widetilde{f}_3(1,2,0,-) = \frac{a[(a+b-m)(\operatorname{tr}(G))^2 + [1+a(b-m)]\operatorname{tr}(G^2)]}{(b-m)[(b-m)^2-1]},$$

$$\begin{aligned}\widetilde{f}_3(1,3,0,-) = \frac{a}{(b-m-2)_5}&[(a+b-m)(2a+b-m)\{(\operatorname{tr}(G))^3 + 2\operatorname{tr}(G^3)\}\\ &+ 3(a+b-m)\{a(b-m)+2\}\operatorname{tr}(G)\operatorname{tr}(G^2)\\ &+ (a^2-1)\{(b-m)^2-4\}\operatorname{tr}(G^3)],\end{aligned}$$

$$\begin{aligned}\widetilde{f}_3(1,2,1,1) = \frac{a}{(a-m-2)_5}&[(a+b-m)\{a(b-m)+2\}\{(\operatorname{tr}(G))^3 + 2\operatorname{tr}(G^3)\}\\ &+ [(a^2+2)\{(b-m)^2+2\} + 9a(b-m)]\operatorname{tr}(G)\operatorname{tr}(G^2)].\end{aligned}$$

Writing trace functions in terms of zonal polynomials, integrating resulting expressions by using (2.2), (2.3) and (2.4), explicit evaluation for $\widetilde{f}_\ell(1,4,0,-)$, $\widetilde{f}_\ell(1,3,1,1)$, $\widetilde{f}_\ell(1,2,1,2)$, $\widetilde{f}_\ell(1,2,2,1)$, $\ell = 1,2,3$, can also be done.

4 Applications

In this section, results derived in Section 3 are used to evaluate expected values of functions of complex beta matrices.

The $m \times m$ random Hermitian positive definite matrix X is said to have a complex matrix variate beta type 1 distribution with parameters $a\,(\geq m)$ and $b\,(\geq m)$, denoted as $X \sim \mathbb{C}\text{B1}(m,a,b)$, if its p.d.f. is given by

$$\{\tilde{B}_m(a,b)\}^{-1}\det(X)^{a-m}\det(I_m - X)^{b-m}, \quad 0 < X^{\mathsf{H}} = X < I_m.$$

By transforming $X = (I_m + Y)^{-1}Y$ with the Jacobian $J(X \to Y) = \det(I_m + Y)^{-2m}$ the p.d.f. of the complex random matrix Y is obtained as

$$\{\tilde{B}_m(a,b)\}^{-1}\det(Y)^{a-m}\det(I_m + Y)^{-(a+b)}, \quad Y^{\mathsf{H}} = Y > 0. \tag{4.5}$$

The random matrix Y is said to have a complex matrix variate beta type 2 distribution with parameters (a,b), denoted as $Y \sim \mathbb{C}\text{B2}(m,a,b)$. The complex matrix variate beta distributions can be derived by using independent complex Wishart matrices. For properties and results the reader is referred to [4], [7], [11], and [12]. The complex matrix-variate beta distributions arise in various problems in multivariate statistical analysis. Several test statistics in multivariate analysis of variance and covariance are functions of beta matrices ([1], [6], [9], and [12]).

If $X \sim \mathbb{C}\text{B1}(m,a,b)$, then by using the p.d.f. of X, we have

$$E[\operatorname{tr}((GX)^2)] = \int_{0<X^{\mathsf{H}}=X<I_m} \frac{\det(X)^{a-m}\det(I_m - X)^{b-m}}{\tilde{B}_m(a,b)} \operatorname{tr}((GX)^2)\,\mathrm{d}X,$$

where G is an $m \times m$ Hermitian matrix. Now, comparing the right hand side of the above expression with that of $\widetilde{f}_1(g,h,i,j)$ in Section 1, we can see that $E[\operatorname{tr}(GX)^2] = \widetilde{f}_1(1,2,0,-)$. Similarly, one can check that $E[\operatorname{tr}(GX)^3] = \widetilde{f}_1(1,3,0,-)$, $E[\operatorname{tr}(GX)^2\operatorname{tr}(GX)] = \widetilde{f}_1(1,2,1,1)$, $E[\operatorname{tr}(X^{-1}G)^2] = \widetilde{f}_2(1,2,0,-)$, $E[\operatorname{tr}(X^{-1}G)^3] = \widetilde{f}_2(1,3,0,-)$, $E[\operatorname{tr}(X^{-1}G)^2\operatorname{tr}(X^{-1}G)] = \widetilde{f}_2(1,2,1,1)$, $E[\operatorname{tr}(GY)^2] = \widetilde{f}_3(1,2,0,-)$, $E[\operatorname{tr}(GY)^3] = \widetilde{f}_3(1,3,0,-)$, $E[\operatorname{tr}(GY)^2\operatorname{tr}(GY)] = \widetilde{f}_3(1,2,1,1)$.

For more results on expected values of functions of complex beta matrices, we refer the readers to [2] and [10].

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