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Approximation Solutions of Backward Fuzzy Stochastic Differential Equations

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Abstract

In this paper, we propose a new formula for backward fuzzy stochastic differential equations (BFSDEs) and prove that the approximation solutions of BFSDEs converge to the exact solution under Lipschitz conditions.

1 Introduction

We are interested in studying the backward fuzzy stochastic differential equations (BFSDEs). We present some research on stochastic differential equations.

Cortes et al. [2] proposed the numerical solution of SDEs using Scheme's random Euler difference method. Nouri and Ranjbar [3] used the Rung-Kutta method to prove approximation solutions for SDE in the presence of initial conditions. Assume the BSDEs with have following

$$X_t = \xi + \int_0^T \Psi(s, X_s)ds - \int_0^T \Gamma_s dW_s \qquad 0 \le t \le T, \tag{1.1}$$

where { W(t),0≤t ≤ T} is a Wiener process on the probability space (Ω, Υ, P) with the filtration $(\Upsilon_t, 0 \le t \le T)$ and ξ is a givenΥ_1-measurable random

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variable so that E$|\xi|^2 < \infty$. Clearly, two random processes $\{$(X(t), Γ(t)), 0 $\leq t \leq T\}$ with values in $R \times R$, which is Υ_t adapted and satisfies the equation (1.1). Such a pair is an adapted solution of the equation. Pardoux and Peng [1] proved the existence and the adapted solutions uniqueness for the Backward SDEs under the Lipschitz condition. Falah and Liu [4] proposed a numerical method for BSDEs with non-Lipschitz coefficients.

On the other hand, Kim [6] proved that the solution for fuzzy stochastic differential equations exists and is unique under the appropriate Lipschitz condition. Malinowski [7] studied fuzzy stochastic integrals and described some of their characteristics. Also, he demonstrated the existence of solutions to SFDEs governed by δ-dimensional Wiener process. In our work, we present some basic concepts and assumptions to study the backward fuzzy SDEs. Moreover, we discuss the approximation solution of BFSDEs under Lipschitz conditions.

2 Preliminaries and Basic Hypotheses

In this section, we offer a few notations and theories that will be applied in the subsequent section. Therefore, we assume that δ-dimensional wiener process $\{W_t\}$,0≤t≤T, for which the entire probability space is defined. $(\Omega, \Upsilon$,P), $\{\Upsilon_t\}_{0\leq t\leq T}$ denotes the filtration with σ-field of Υ_t-progressively subsets of $\Omega \times [0,T]$. The following spaces are used:

1- Let $H^2_T(R^\lambda)$ be the space of Υ-adapted processes X: $\omega\times$[0,T]$\longrightarrow R^\lambda$ such that $E[\sup_{0\leq t\leq T}|X(t)|^2] < \infty$.

2-Let $S^2_T(R^{\lambda\times\delta})$ be the space of Υ-predictable processes $\Gamma : \omega \times [0,T] \longrightarrow R^{\lambda\times\delta}$ such that $E\int_0^T|\Gamma(t)|^2$dt$< \infty$.

The spaces $H^2_T(R^\lambda)$ and $S^2_T(R^{\lambda\times\delta})$ are equipped with the norms $||X||^2_{H^2_T}$=$E[\sup_{0\leq t\leq T}|X(t)|^2]$ and $||\Gamma||^2_{S^2_T}$ =E$\int_0^T|\Gamma(t)|^2$dt, respectively.
We Assume the formula of the backward stochastic differential equation:

$$X_t = \xi + \int_{t_0}^T [\Psi(s, X(s), \Gamma(s))]ds + \int_{t_0}^T [\Phi(s, X(s)) - \Gamma(s)]dWs, \qquad (2.2)$$

where Ψ and Φ are Borel measurable function, $\{W(t), 0 \leq t \leq T\}$ is a δ-dimensional Wiener Process, and xi is given Υ_t-measurable random variable combined with $E|\xi|^2 < \infty$.

Suppose that $\Psi : \Omega \times [0,T] \times R^{\lambda} \times R^{\lambda\times\delta} \longrightarrow R^{\lambda}$ and $\Phi : \Omega \times [0,T] \times R^{\lambda} \longrightarrow R^{\lambda\times\delta}$ are jointly measurable and $(X,\Gamma) \in R^{\lambda} \times R^{\lambda\times\delta}$.

The following conditions are used:

$$(H_1) \qquad |\Psi(X_1,\Gamma_1,t) - \Psi(X_2,\Gamma_2,t)|^2 \leq K(|X_1 - X_2|^2 + |\Gamma_1 - \Gamma_2|^2) a.s$$
$$|\Phi(X_1,t) - \Phi(X_2,t)|^2 \leq K(|X_1 - X_2|^2),$$

where $K > 0$, and for all $t \in [0,T], X_1, X_2 \in R^{\lambda}$ and $\Gamma_1, \Gamma_2 \in R^{\lambda\times\delta}$

$$(H_2) \qquad |\Psi(X,\Gamma)|^2 \vee |\Phi(X,\Gamma)|^2 \leq K(1 + |X|^2 + |\Gamma|^2),$$

where $c \vee d = \max\{c,d\}$.

$$(H_3) \qquad E|\xi(t) - \xi(s)|^2 \leq K(t-s).$$

3 Fuzzy Solution of the BSDEs

Let $\eta(R^{\lambda})$ be the family of all compact, convex, nonempty subsets R^{λ}. We use the fuzzy set space of to denote R^{λ} as $B(R^{\lambda})$; i.e., the set of functions $\varphi : R^{\lambda} \longrightarrow [0,1]$ such that $[\varphi]^{\beta} \in \eta(R^{\lambda})$ for every $\beta \in [0,1]$, where $[\varphi]^{\beta} = \{a \in R^{\lambda} : \varphi(a) \leq \beta\}$ for $\beta \in [0,1]$ and $[\varphi]^0 = \{a \in R^{\lambda} : \varphi(a) > 0\}$. Assume (Ω, Υ, P) is a probability space together with a filtration $\{\Upsilon_t\}_{t\in[0,T]}, T \in (0,\infty)$ under the usual conditions. A mapping $X : \Omega \longrightarrow B(R^{\lambda})$ is called fuzzy random variable, if $[X]^{\beta} : \Omega \longrightarrow \eta(R^{\lambda})$ is an Υ-measurable multifunction every $\beta \in [0,1]$.

Definition 3.1. *[7] Assume (Ω, Υ, P) is a complete probability space. We say $\mathcal{Y} : \Omega \longrightarrow \Upsilon(R^{\lambda})$ is a fuzzy random variable if for each $\beta \in$[0,1], the mapping $[X]^{\beta} : \Omega \longrightarrow \eta(R^{\lambda})$ is Υ-measurable.*

Definition 3.2. *[7] A fuzzy stochastic process is a mapping $X : [0,1] \times \Omega \longrightarrow \Upsilon(R_{\Lambda})$, where $X(t,\cdot)$ or $X(t) : \Omega \longrightarrow \Upsilon(R_{\lambda})$ is a fuzzy random variable.*

Definition 3.3. *[7] If the functions $X(.,W) : [0,1] \longrightarrow \Upsilon(R^{\lambda})$ are δ_{∞}-continuous mappings, then a fuzzy stochastic process X is considered to be δ_{∞}-continuous.*

We consider the BFSDEs as
$dX(t) = \Psi(t, X(t), \Gamma(t))dt + [\Phi(t, X(t)) - \Gamma(t)]dW(t)$
$X(T) = \xi_T =' \xi(X(T), \Gamma(T))$, where $\Psi : \Omega \times [0,1] \times R^{\lambda} \times R^{\lambda\times\delta} \times A(R^{\lambda}) \longrightarrow A(R^{\lambda})$ and $X_T : \Omega \longrightarrow A(R^{\lambda})$ is a fuzzy random variable.

4 Numerical Formula for BFSDEs

In this section, we propose a numerical formula that is based on a decretization of (1.1). Therefore, let $n \geq 1$ and $t \in [0,T]$. Assume $0 = t_0 < t_1 < .... < t_n = T$ is a partition of $[0,T]$. Let

$$\pi = \triangle t_{i+1} = t_{i+1} - t_i = \tfrac{T}{n}, 1 \leq i \leq n, \triangle W_{t_{i+1}} = W_{t_{i+1}} - W_{t_i},$$

where $i = 0, 1, ..., n-1$ and $\triangle t = \max \triangle t_i$. For the small interval $[t_i, t_{i+1}]$, the BFSDEs are as follows:

$$X_{t_i} = X_{t_{i+1}} + \int_{t_i}^{t_{i+1}} [\Psi(s, X(s), \Gamma(s))] ds + \int_{t_i}^{t_{i+1}} [\Phi(s, X(s)) - \Gamma(s)] dW(s). \tag{4.3}$$

Therefore, the approximation formula is:

$$X^n_{t_i} = X^n_{t_{i+1}} + \Psi(t, X^n_i(t), \Gamma^n_i(t))\pi + \Phi(t, X^n_i(t)) - \Gamma^n_i(t) \triangle W_{t_{i+1}},$$

with $X(T) = \xi(T)$ on $0 \leq t \leq T$. Thus we consider the BFSDEs as follows:

$$X^n_{t_i} = \xi + \int_{t_0}^{T} [\Psi(s, X^n_i(s), \Gamma^n_i(s))] ds + \int_{t_0}^{T} [\Phi(s, X^n_i(s)) - \Gamma^n_i(s)] dW(s) \tag{4.4}$$

5 Main results

The purpose of this section is to discuss approximation solutions to the BFSDE equations.

Theorem 5.1. *Assume that* $\{X(t), \Gamma(t)\}$ *is a solution of equation* (2.2). *Then the approximation solution* $\{X^n(t), \Gamma^n(t)\}$ *converges to* $\{X(t), \Gamma(t)\}$ *in the sense that for each* $t \in [0,T]$

$$\lim_{n \longrightarrow \infty} E|X(t) - X^n(t)|^2 = 0$$

and

$$\lim_{n \longrightarrow \infty} E \int_0^T |\Gamma(s) - \Gamma^n(s)|^2 ds = 0$$

Proof.

$$|X(t) - X^n(t)|^2 = |\xi - \xi^n + \int_0^T [\Psi(s, X(s), \Gamma(s)) - \Psi(s, X^n(s), \Gamma^n(s))] ds$$
$$+ \int_0^T [(\Phi(s, X(s)) - \Gamma(s)) - (\Phi(s, X^n(s)) - \Gamma^n(s))] dWs|^2.$$

Using the inequality $\{|d+e+h|^2 \leq 3(|d|^2+|e|^2+|h|^2)\}$, we have

$$|X(t)-X^n(t)|^2 \leq 3|X(T)-X^n(T)|^2 + 3(T)\int_0^T [|\Psi(s,X(s),\Gamma(s))-\Psi(s,X^n(s),\Gamma^n(s))|^2]ds$$
$$+3\int_0^T [|(\Phi(s,X(s))-\Phi(s,X^n(s))|^2+|\Gamma(s)-\Gamma^n(s)|^2]|dWs|^2.$$

Using condition (H_1) and (H_3), we have

$$E|X(t)-X^n(t)|^2 \leq 3E|X(T)-X^n(T)|^2 + 3K(T)E\int_0^T [|X(s)-X^n(s)|^2$$
$$+|\Gamma(s)-\Gamma^n(s)|^2]ds + 3KE\int_0^T [|X(s)-X^n(s))|^2+|\Gamma(s)-\Gamma^n(s)|^2]ds.$$

$$E|X(t)-X^n(t)|^2 \leq 3K(T-0)+3K(T+1)\int_0^T E[|X(s)-X^n(s)|^2+|\Gamma(s)-\Gamma^n(s)|^2]ds.$$

Let $C_1 = 3K(T)$, $C_2 = 3K(T+1)$. We obtain

$$E|X(t)-X^n(t)|^2 \leq C_1 + C_2\int_0^T E|X(s)-X^n(s)|^2 + E|\Gamma(s)-\Gamma^n(s)|^2 ds.$$

Next,

$$E|X(t)-X^n(t)|^2 \leq C_1 + C_2\int_0^T E|X(s)-X^n(s)|^2 ds,$$

for all $t \in [0,T]$. Using theorem (1.8.1) (Gronwall's inequality) in [9], we have

$$\lim_{n\longrightarrow\infty} E|X(t)-X^n(t)|^2 = 0.$$

So

$$\lim_{n\longrightarrow\infty} E\int_0^T |\Gamma(s)-\Gamma^n(s)|^2 ds = 0.$$

Theorem 5.2. *Suppose that the assumptions $H_1 - H_3$ are fulfilled. Then there exists a unique solution of the following equation:*

$$X_t = \xi + \int_0^T [\Psi(s,X(s),\Gamma(s))]ds + \int_0^T [\Phi(s,X(s))-\Gamma(s)]dWs.$$

Proof. Existence. By theorem (4.2) in [5], there exist

$$X \in H_T^2(R^\lambda) \quad and \quad \Gamma \in S_T^2(R^{\lambda\times\delta})$$

such that $\lim_{n\longrightarrow\infty}(X^n,\Gamma^n) = (X,\Gamma)$. Then lemma (3.1) in [8] and theorem (5.1) show that

$$\lim_{n\longrightarrow\infty} E|X^n(t) - X(t)|^2 = 0, \qquad 0 \le t \le T.$$

The result now follows by lemma (3.1) in [8], lemma (3.2) in [8], and theorem (4.2) in [5].

Uniqueness. Consider the pair (X_i,Γ_i), where $i = 1, 2$ representing the solutions of BFSDEs.
By theorem (5.1), we have
$\lim_{n\longrightarrow\infty} |X(t) - X_1^n(t)|^2 = 0$ and $\lim_{n\longrightarrow\infty} |X(t) - X_2^n(t)|^2 = 0$.
Now, we prove

$$\lim_{n\longrightarrow\infty} |X_1^n(t) - X_2^n(t)|^2 = 0.$$

$$\begin{aligned}|X_1^n(t) - X_2^n(t)|^2 =& |\xi_1^n - \xi_2^n + \int_0^T [\Psi(s, X_1^n(s), \Gamma_1^n(s)) - \Psi(s, X_2^n(s), \Gamma_2^n(s))]ds \\ &+ \int_0^T (\Phi(s, X_1^n(s)) - \Gamma_1^n(s)) - (\Phi(s, X_2^n(s)) - \Gamma_2^n(s))dWs|^2.\end{aligned}$$

Using the inequality $|d + e + h|^2 \le 3(|d|^2 + |e|^2 + |h|^2)$, we have

$$\begin{aligned}|X_1^n(t) - X_2^n(t)|^2 \le& 3|X_1^n(T) - X_2^n(T|^2 + 3|\int_0^T [\Psi(s, X_1^n(s), \Gamma_1^n(s)) - \Psi(s, X_2^n(s), \Gamma_2^n(s))]ds|^2 \\ &+ 3|\int_0^T (\Phi(s, X_1^n(s)) - \Phi(s, X_2^n(s)) - (\Gamma_1^n(s) - \Gamma_2^n(s))dWs|^2.\end{aligned}$$

Using conditions H_1 and H_3, we obtain

$$\begin{aligned}E|X_1^n(t) - X_2^n(t)|^2 \le& 3K(T) + 3K(T)\int_0^T E(|X_1^n(s) - X_2^n(s)|^2 + |\Gamma_1^n(s) - \Gamma_2^n(s)|^2)ds \\ &+ 3K\int_0^T E(|X_1^n(s) - X_2^n(s)|^2 + |\Gamma_1^n(s) - \Gamma_2^n(s)|^2 ds.\end{aligned}$$

Let $C_1 = 3K(T)$, $C_2 = 3K(T + 1)$. We have

$$E|X_1^n(t) - X_2^n(t)|^2 \le C_1 + C_2\int_0^T E|X_1^n(s) - X_2^n(s)|^2 + E|\Gamma_1^n(s) - \Gamma_2^n(s)|^2 ds.$$

Then $E|X_1^n(t) - X_2^n(t)|^2 \le C_1 + C_2 \int_0^T E|X_1^n(s) - X_2^n(s)|^2$,
for all $t \in [0, T]$.

Using theorem (1.8.1) (Gronwall's inequality) in [9], we have

$$\lim_{n \longrightarrow \infty} E|X_1^n(t) - X_2^n(t)|^2 = 0.$$

Hence $X_1^n(t) = X_2^n(t)$.
As a result, we get $\lim_{n\longrightarrow\infty} E|\Gamma_1^n(t) - \Gamma_2^n(t)|^2 = 0$. □

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