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The Egyptian fraction of the form $\frac{1}{a} + \frac{1}{b} = \frac{q-1}{pq}$

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Abstract

We determine all positive integer solutions of the equation $\frac{1}{a} + \frac{1}{b} = \frac{q-1}{pq}$, where p and q are prime numbers with $p > q$.

1 Introduction

Representing a rational as a sum of positive distinct unit fractions is called an Egyptian fraction. Many mathematicians find the Egyptian fractions very interesting. In 2022, Johnson [1] solved the general equation

$$\frac{1}{a} + \frac{1}{b} = \frac{q+1}{pq}$$

where p and q are distinct primes such that $q+1 \mid p-1$ which appeared back in the 2018 William Lowell Putnam Mathematical Competition [2]. We wish to explore an equation similar to the above equation; namely,

$$\frac{1}{a} + \frac{1}{b} = \frac{q-1}{pq}, \tag{1.1}$$

where p and q are prime numbers with $p > q$.

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2 Main results

Let's rewrite equation (1.1) as $(a+b)pq = (q-1)ab$. Since $p > q$ are primes, we must have the case that p divides both a and b or p divides a or b. Suppose first that p divides both a and b. Write $a = px$ and $b = py$, for some positive integers x and y. Then $(x+y)q = (q-1)xy$. Since q and $q-1$ are relatively prime, q divides x or q divides y. Without loss of generality, we assume that q divides x. Then we write $x = qx'$, for some positive integer x'. Thus

$$y(1+x') = qx'(y-1).$$

Since $\gcd(x', 1+x') = \gcd(y-1, y) = 1$, $y = x'$ or $y = q$ or $y = qx'$. Suppose $y = x'$. Then $1 + x' = q(x'-1)$ and this implies that $q = 1 + \frac{2}{x'-1}$. Thus $x' = 2$ or 3 which implies that $q = 3$ or $q = 2$, respectively. Hence we obtain two solutions; namely, $(a, b, p, q) = (6p, 2p, p, 3)$, where $p > 3$ is prime and $(a, b, p, q) = (6p, 3p, p, 2)$, where $p > 2$ is prime.

Next, suppose that $y = q$. We have $x' = 1$ and we obtain a positive solution $(3p, 3p, p, 3)$, where $p > 3$ is a prime.

Now consider the case $y = q'x$. Then $q = 1 + \frac{2}{x'}$. Thus $x' = 1$ or 2 and this implies that $q = 3$ or 2. Hence we obtain two solutions; namely, $(a, b, p, q) = (3p, 3p, p, 3)$, where $p > 3$ is prime and $(a, b, p, q) = (4p, 4p, p, 2)$, where $p > 2$ is prime.

Now suppose p divides a but p does not divide b. Write $a = px$, for some positive integer x. Then

$$pxq = b((q-1)x - q).$$

Since $\gcd(q-1, q) = 1$, we have q divides b or q divides x.

If q divides b, then we write $b = qy$, for some positive integer y. Then

$$qy + px = (q-1)xy.$$

Obviously, x divides qy. If q and x are relatively prime, then x divides y and we write $y = xy'$, for some positive integer y'. Thus

$$p = y'(qx - x - q).$$

Since p is relatively prime to y', we have $y' = 1$. This implies that $x = y$ and $p = qx - x - q$. Thus $q = (p+x)/(x-1)$. Therefore, the positive solution is

$$(a, b, p, q) = (px, qx, p, \frac{p+x}{x-1}),$$

where $(p+x)/(x-1)$ is a prime.

If q divides x, then we write $x = qx'$, for some positive integer x'. We have

$$px'q = b((q-1)x'-1).$$

Since p does not divide b, p divides $(q-1)x'-1$. Thus

$$x'q = b\left(\frac{(q-1)x'-1}{p}\right).$$

Since $\gcd(x', (q-1)x'-1)$, there are two possible cases as follows:

Case 1 $x' = b$. We have $a = pqb$, where $b = (pq+1)/(q-1)$. It is easy to see that $q-1$ divides $p+1$ if and only if $q-1$ divides $pq+1$. Thus we obtain a new solution

$$\left(\frac{pq(1+pq)}{q-1}, \frac{1+pq}{q-1}, p, q\right),$$

where $q-1$ divides $p+1$.

Case 2 $x' = 1$. Thus $q = b(q-2)/p$ is not an integer because p does not divide b and $p > q$.

In conclusion, we have proved the following theorem:

Theorem 2.1. *Let $p > q$ be primes. The positive integer solutions of the Diophantine equation*

$$\frac{1}{a}+\frac{1}{b}=\frac{q-1}{pq}$$

are:

1. $(a,b,p,q) = (6p,2p,p,3), (6p,3p,p,2), (3p,3p,p,3), (4p,4p,p,2)$,
2. $(a,b,p,q) = (px,qx,p,q)$, *where* $x = (p+q)/(q-1)$ *is a positive integer,*
3. $(a,b,p,q) = ((pq(1+pq))/(q-1), (1+pq)/(q-1), p, q)$, *where* $q-1$ *divides* $p+1$.

References

[1] Jeremiah W. Johnson, A Diophantine Equation with an Elementary Solution *Coll. Math. J.,* **53,** (2022), 361–363.

[2] The William Lowell Putnam Mathematical Competition, (2019). Available at: *https://www.maa.org/sites/ default/files/pdf/Putnam/Competition Archive/2018PutnamProblems.pdf.* Accessed December 28, 2022.