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On $\delta s(\Lambda, s)$-open sets in topological spaces

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Abstract

Our main purpose is to introduce the concept of $\delta s(\Lambda, s)$-open sets. Moreover, we investigate some properties of $\delta s(\Lambda, s)$-open sets and $\delta s(\Lambda, s)$-closed sets.

1 Introduction

Semi-open sets, preopen sets, α-open sets, β-open sets and δ-open sets play an important role in the theory of classical point set topology. In 1963, Levine [5] introduced the concept of semi-open sets which is weaker than the concept of open sets in topological spaces. Veličko [8] introduced δ-open sets, which are stronger than open sets. Park et al. [6] have offered a new notion called δ-semiopen sets which are stronger than semi-open sets but weaker than δ-open sets and investigated the relationships between several types of these open sets. Caldas et al. [3] investigated some weak separation axioms by utilizing δ-semiopen sets and the δ-semiclosure operator. Caldas et al. [2] investigated the notion of δ-Λ_s-semiclosed sets which is defined as the intersection of a δ-Λ_s-set and a δ-semiclosed set. In [1], the present authors introduced and investigated the concept of (Λ, s)-closed sets by utilizing the

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notions of Λ_s-sets and semi-closed sets. In this paper, we introduce the concept of $\delta s(\Lambda, s)$-open sets. In particular, some properties of $\delta s(\Lambda, s)$-open sets and $\delta s(\Lambda, s)$-closed sets are discussed.

2 Preliminaries

Throughout the present paper, unless explicitly stated spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed . Let A be a subset of a topological space (X, τ). The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A of a topological space (X, τ) is called *semi-open* [5] if $A \subseteq \mathrm{Cl}(\mathrm{Int}(A))$. The family of all semi-open sets in a topological space (X, τ) is denoted by $SO(X, \tau)$. A subset A^{Λ_s} [4] is defined as follows: $A^{\Lambda_s} = \cap\{U \mid U \supseteq A, U \in SO(X, \tau)\}$. A subset A of a topological space (X, τ) is called a Λ_s-*set* [4] if $A = A^{\Lambda_s}$. A subset A of a topological space (X, τ) is called (Λ, s)-*closed* [1] if $A = T \cap C$, where T is a Λ_s-set and C is a semi-closed set. The complement of a (Λ, s)-closed set is called (Λ, s)-*open*. The family of all (Λ, s)-closed (resp. (Λ, s)-open) sets in a topological space (X, τ) is denoted by $\Lambda_s C(X, \tau)$ (resp. $\Lambda_s O(X, \tau)$). Let A be a subset of a topological space (X, τ). A point $x \in X$ is called a (Λ, s)-*cluster point* [1] of A if for every (Λ, s)-open set U of X containing x we have $A \cap U \neq \emptyset$. The set of all (Λ, s)-cluster points of A is called the (Λ, s)-*closure* [1] of A and is denoted by $A^{(\Lambda,s)}$. The union of all (Λ, s)-open sets contained in A is called the (Λ, s)-*interior* [1] of A and is denoted by $A_{(\Lambda,s)}$. Let A be a subset of a topological space (X, τ). A point x of X is called a $\delta(\Lambda, s)$-*cluster point* [7] of A if $A \cap [V^{(\Lambda,s)}]_{(\Lambda,s)} \neq \emptyset$ for every (Λ, s)-open set V of X containing x. The set of all $\delta(\Lambda, s)$-cluster points of A is called the $\delta(\Lambda, s)$-*closure* [7] of A and is denoted by $A^{\delta(\Lambda,s)}$. If $A = A^{\delta(\Lambda,s)}$, then A is said to be $\delta(\Lambda, s)$-*closed* [7]. The complement of a $\delta(\Lambda, s)$-closed set is said to be $\delta(\Lambda, s)$-*open* [7]. The union of all $\delta(\Lambda, s)$-open sets contained in A is called the $\delta(\Lambda, s)$-*interior* [7] of A and is denoted by $A_{\delta(\Lambda,s)}$.

3 $\delta s(\Lambda, s)$-open sets

We begin this section by introducing the notion of $\delta s(\Lambda, s)$-open sets.

Definition 3.1. *A subset A of a topological space (X, τ) is said to be $\delta s(\Lambda, s)$-open if $A \subseteq [A_{(\Lambda,s)}]^{\delta(\Lambda,s)}$. The complement of a $\delta s(\Lambda, s)$-open set is said to be $\delta s(\Lambda, s)$-closed.*

The family of all $\delta s(\Lambda, s)$-open (resp. $\delta s(\Lambda, s)$-closed) sets in a topological space (X, τ) is denoted by $\delta s(\Lambda, s)O(X, \tau)$ (resp. $\delta s(\Lambda, s)C(X, \tau)$). A subset N of a topological space (X, τ) is called a $\delta s(\Lambda, s)$-*neighborhood* of a point $x \in X$ if there exists a $\delta s(\Lambda, s)$-open set V such that $x \in V \subseteq N$.

Definition 3.2. *Let A be a subset of a topological space (X, τ). A point x of X is called a $\delta s(\Lambda, s)$-cluster point of A if $A \cap U \neq \emptyset$ for every $\delta s(\Lambda, s)$-open set U of X containing x. The set of all $\delta s(\Lambda, s)$-cluster points of A is called the $\delta s(\Lambda, s)$-closure of A and is denoted by $A^{\delta s(\Lambda, s)}$.*

Lemma 3.3. *The intersection of an arbitrary collection of $\delta s(\Lambda, s)$-closed sets in X is $\delta s(\Lambda, s)$-closed.*

Corollary 3.4. *Let A be a subset of a topological space (X, τ). Then $A^{\delta s(\Lambda, s)} = \cap\{F \in \delta s(\Lambda, s)C(X, \tau) \mid A \subseteq F\}$.*

Lemma 3.5. *For the $\delta s(\Lambda, s)$-closure of subsets A and B in a topological space (X, τ), the following properties hold:*

(1) A is $\delta s(\Lambda, s)$-closed in (X, τ) if and only if $A = A^{\delta s(\Lambda, s)}$.

(2) If $A \subseteq B$, then $A^{\delta s(\Lambda, s)} \subseteq B^{\delta s(\Lambda, s)}$.

(3) $A^{\delta s(\Lambda, s)}$ is $\delta s(\Lambda, s)$-closed; that is, $A^{\delta s(\Lambda, s)} = [A^{\delta s(\Lambda, s)}]^{\delta s(\Lambda, s)}$.

Definition 3.6. *Let A be a subset of a topological space (X, τ). A subset $\delta s(\Lambda, s)Ker(A)$ is defined as follows:*

$$\delta s(\Lambda, s)Ker(A) = \cap\{U \mid A \subseteq U, U \in \delta s(\Lambda, s)O(X, \tau)\}.$$

Lemma 3.7. *For subsets A and B of a topological space (X, τ), the following properties hold:*

(1) $A \subseteq \delta s(\Lambda, s)Ker(A)$.

(2) If $A \subseteq B$, then $\delta s(\Lambda, s)Ker(A) \subseteq \delta s(\Lambda, s)Ker(B)$.

(3) $\delta s(\Lambda, s)Ker(\delta s(\Lambda, s)Ker(A)) = \delta s(\Lambda, s)Ker(A)$.

(4) If A is $\delta s(\Lambda, s)$-open, $\delta s(\Lambda, s)Ker(A) = A$.

Definition 3.8. *Let (X, τ) be a topological space and $x \in X$. A subset $\langle x\rangle_{\delta s(\Lambda, s)}$ is defined as follows: $\langle x\rangle_{\delta s(\Lambda, s)} = \delta s(\Lambda, s)Ker(\{x\}) \cap \{x\}^{\delta s(\Lambda, s)}$.*

Theorem 3.9. *For a topological space (X, τ), the following properties hold:*

(1) $\delta s(\Lambda, s)Ker(A) = \{x \in X \mid A \cap \{x\}^{\delta s(\Lambda,s)} \neq \emptyset\}$ *for each subset* A *of* X.

(2) For each $x \in X$, $\delta s(\Lambda, s)Ker(\langle x\rangle_{\delta s(\Lambda,s)}) = \delta s(\Lambda, s)Ker(\{x\})$.

(3) For each $x \in X$, $[\langle x\rangle_{\delta s(\Lambda,s)}]^{\delta s(\Lambda,s)} = \{x\}^{\delta s(\Lambda,s)}$.

(4) If U *is* $\delta s(\Lambda, s)$*-open in* X *and* $x \in U$, *then* $\langle x\rangle_{\delta s(\Lambda,s)} \subseteq U$.

(5) If F *is* $\delta s(\Lambda, s)$*-closed in* X *and* $x \in F$, *then* $\langle x\rangle_{\delta s(\Lambda,s)} \subseteq F$.

Proof. (1) Suppose that $A \cap \{x\}^{\delta s(\Lambda,s)} = \emptyset$. Then $x \notin X - \{x\}^{\delta s(\Lambda,s)}$ which is a $\delta s(\Lambda, s)$-open set containing A. Thus $x \notin \delta s(\Lambda, s)Ker(A)$ and hence

$$\delta s(\Lambda, s)Ker(A) \subseteq \{x \in X \mid A \cap \{x\}^{\delta s(\Lambda,s)} \neq \emptyset\}.$$

Next, let $x \in X$ such that $A \cap \{x\}^{\delta s(\Lambda,s)} \neq \emptyset$. Suppose that $x \notin \delta s(\Lambda, s)Ker(A)$. Then there exists a $\delta s(\Lambda, s)$-open set U containing A and $x \notin U$. Let $y \in A \cap \{x\}^{\delta s(\Lambda,s)}$. Therefore, U is a $\delta s(\Lambda, s)$-neighbourhood of y which does not contain x. This contradiction leads to $x \in \delta s(\Lambda, s)Ker(A)$.

(2) Let $x \in X$. Then $\{x\} \subseteq \{x\}^{\delta s(\Lambda,s)} \cap \delta s(\Lambda, s)Ker(\{x\}) = \langle x\rangle_{\delta s(\Lambda,s)}$. By Lemma 3.7, $\delta s(\Lambda, s)Ker(\{x\}) \subseteq \delta s(\Lambda, s)Ker(\langle x\rangle_{\delta s(\Lambda,s)})$. Next, we show the opposite implication. Suppose that $y \notin \delta s(\Lambda, s)Ker(\{x\})$. Then there exists a $\delta s(\Lambda, s)$-open set V such that $x \in V$ and $y \notin V$. Since $\langle x\rangle_{\delta s(\Lambda,s)} \subseteq \delta s(\Lambda, s)Ker(\{x\}) \subseteq \delta s(\Lambda, s)Ker(V) = V$, we have $\delta s(\Lambda, s)Ker(\langle x\rangle_{\delta s(\Lambda,s)}) \subseteq V$. Since $y \notin V$, $y \notin \delta s(\Lambda, s)Ker(\langle x\rangle_{\delta s(\Lambda,s)})$ and hence $\delta s(\Lambda, s)Ker(\langle x\rangle_{\delta s(\Lambda,s)}) \subseteq \delta s(\Lambda, s)Ker(\{x\})$. Thus $\delta s(\Lambda, s)Ker(\{x\}) = \delta s(\Lambda, s)Ker(\langle x\rangle_{\delta s(\Lambda,s)})$.

(3) By the definition of $\langle x\rangle_{\delta s(\Lambda,s)}$, we have $\{x\} \subseteq \langle x\rangle_{\delta s(\Lambda,s)}$ and $\{x\}^{\delta s(\Lambda,s)} \subseteq [\langle x\rangle_{\delta s(\Lambda,s)}]^{\delta s(\Lambda,s)}$ by Lemma 3.5. On the other hand, we have $\langle x\rangle_{\delta s(\Lambda,s)} \subseteq \{x\}^{\delta s(\Lambda,s)}$ and $(\langle x\rangle_{\delta s(\Lambda,s)})^{\delta s(\Lambda,s)} \subseteq (\{x\}^{\delta s(\Lambda,s)})^{\delta s(\Lambda,s)} = \{x\}^{\delta s(\Lambda,s)}$. Therefore, $(\langle x\rangle_{\delta s(\Lambda,s)})^{\delta s(\Lambda,s)} \subseteq \{x\}^{\delta s(\Lambda,s)}$.

The proofs of (4) and (5) are straightforward. □

Theorem 3.10. *For any points* x *and* y *in a topological space* (X, τ), *the following properties are equivalent:*

(1) $\delta s(\Lambda, s)Ker(\{x\}) \neq \delta s(\Lambda, s)Ker(\{y\})$.

(2) $\{x\}^{\delta s(\Lambda,s)} \neq \{y\}^{\delta s(\Lambda,s)}$.

Proof. (1) $\Rightarrow$ (2): Suppose that $\delta s(\Lambda, s)Ker(\{x\}) \neq \delta s(\Lambda, s)Ker(\{y\})$. There exists a point $z \in X$ such that $z \in \delta s(\Lambda, s)Ker(\{x\})$ and $z \notin \delta s(\Lambda, s)Ker(\{y\})$ or $z \in \delta s(\Lambda, s)Ker(\{y\})$ and $z \notin \delta s(\Lambda, s)Ker(\{x\})$. We prove only the first case with the second case being analogous. From $z \in \delta s(\Lambda, s)Ker(\{x\})$, it follows that $\{x\} \cap \{z\}^{\delta s(\Lambda,s)} \neq \emptyset$ which implies $x \in \{z\}^{\delta s(\Lambda,s)}$. By $z \notin \delta s(\Lambda, s)Ker(\{y\})$, we have $\{y\} \cap \{z\}^{\delta s(\Lambda,s)} = \emptyset$. Since $x \in \{z\}^{\delta s(\Lambda,s)}$, we have $\{x\}^{\delta p(\Lambda,s)} \subseteq \{z\}^{\delta s(\Lambda,s)}$ and $\{y\} \cap \{x\}^{\delta s(\Lambda,s)} = \emptyset$. Therefore, $\{x\}^{\delta s(\Lambda,s)} \neq \{y\}^{\delta s(\Lambda,s)}$. Thus $\delta s(\Lambda, s)Ker(\{x\}) \neq \delta s(\Lambda, s)Ker(\{y\})$ and hence $\{x\}^{\delta s(\Lambda,s)} \neq \{y\}^{\delta s(\Lambda,s)}$.

(2) $\Rightarrow$ (1): Suppose that $\{x\}^{\delta s(\Lambda,s)} \neq \{y\}^{\delta s(\Lambda,s)}$. There exists a point $z \in X$ such that $z \in \{x\}^{\delta s(\Lambda,s)}$ and $z \notin \{y\}^{\delta s(\Lambda,s)}$ or $z \in \{y\}^{\delta s(\Lambda,s)}$ and $z \notin \{x\}^{\delta s(\Lambda,s)}$. We prove only the first case with the second case being analogous. It follows that there exists a $\delta s(\Lambda, s)$-open set containing z and therefore x but not y; namely, $y \notin \delta s(\Lambda, s)Ker(\{x\})$ and thus $\delta s(\Lambda, s)Ker(\{x\}) \neq \delta s(\Lambda, s)Ker(\{y\})$. $\square$

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