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Clique Connected Common Neighborhood Polynomial of the Join of Graphs

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Abstract

In this paper, we establish the clique connected common neighborhood polynomial of the graph resulting from the join of two connected graphs.

1 Introduction

The study of graph polynomials captured the interests of several mathematicians in recent years because of its widespread applications in Chemistry [8],

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Biology, and Physics [5]. These graph polynomials included the clique polynomials and independent sets polynomials by Hoede and Li [10] in 1994, and the neighborhood polynomial by Brown and Nowakowski [4] in 2008. The above polynomials are univariate. In 2022, Artes, Langamin, and Calig-og [2] introduced a bivariate graph polynomial called the clique common neighborhood polynomial of a graph by considering the common neighborhood system of a clique in a graph. This bivariate polynomial generalizes the clique polynomial defined in [10].

Graphs considered in this study are simple, connected, and undirected. For standard terminologies and notations, we refer the readers to [3, 9, 7].

If $v \in V(G)$, the *open neighborhood* or simply the *neighborhood* of v in G is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. The *closed neighborhood* of v in G is the set $N_G[v] = \{v\} \cup N_G(v)$. For a subset S of $V(G)$, the *neighborhood system* of S in G is the set $N_G(S) = \bigcup_{s\in S} N_G(s) \backslash S$. The *common neighborhood system* of S in G, denoted by $N_G^c(S)$, is the set $N_G^c(S) = \bigcap_{s\in S} N_G(s)$.

Given a nontrivial simple connected graph G, the *clique common neighborhood polynomial* of G is given by

$$ccn(G; x, y) = \sum_{j=0}^{n-i} \sum_{i=1}^{\omega(G)} c_{ij}(G) x^i y^j, \tag{1.1}$$

where $c_{ij}(G)$ is the number of i-cliques in G with common neighborhood cardinality equal to j and $\omega(G)$ is the cardinality of a maximum clique in G, called the *clique number* of G. This was first defined in [2].

In our paper [1], we extended the idea of the clique common neighborhood polynomial of a graph into a more restricted case by considering the maximum connected subset of the common neighborhood system of a clique in a graph. We introduced a new graph polynomial as follows: The *clique connected common neighborhood polynomial* of a graph G, denoted by $\Omega_{cccn}(G; x, y)$, is given by

$$\Omega_{cccn}(G; x, y) = \sum_{j=0}^{n-i} \sum_{i=1}^{\omega(G)} \omega_{ij}(G) x^i y^j, \tag{1.2}$$

where $\omega_{ij}(G)$ is the number of i-cliques in G with a maximum connected subset of the common neighborhood system has cardinality equal to j, and $\omega(G)$ is the cardinality of a maximum clique in G, called the *clique number* of G. In

[1], we established the clique connected common neighborhood polynomials of the complete, complete bipartite, and complete q-partite graphs.

In this paper, we will characterize the cliques in the join $G \oplus H$ of two connected graphs G and H, and establish the clique connected common neighborhood polynomial of $G \oplus H$.

2 Results

In this section, we present the main result of the study. First, we characterize the cliques in the join $G \oplus H$ of two connected graphs G and H. The characterization is our basis in order to establish the clique connected common neighborhood polynomial of $G \oplus H$ by considering the maximum connected subset of the common neighborhood systems of cliques in $G \oplus H$.

The join of two graphs G and H is the graph $G \oplus H$ with vertex-set $V(G \oplus H) = V(G) \cup V(H)$ and order $|V(G \oplus H)| = |V(G)| + |V(H)|$. The edge-set of $G \oplus H$ is $E(G \oplus H) = E(G) \cup E(H) \cup \{u, v : u \in V(G), v \in V(H)\}$ and its size is $|E(G \oplus H)| = |E(G)| + |E(H) + |V(G)||V(H)|$.

The following lemma follows easily from the definition of the join of graphs.

Lemma 2.1. *Let $S \subseteq V(G \oplus H)$. If S intersects G and H, then S is connected in $G \oplus H$.*

The following lemma characterizes the cliques in the join $G \oplus H$ of two connected graphs G and H.

Lemma 2.2. *A subset S of $V(G \oplus H)$ is a clique in $G \oplus H$ if and only if it satisfies one of the following conditions:*

(i) *S is a clique in G*

(ii) *S is a clique in H*

(iii) *$S = S_G \cup S_H$, where S_G is a clique in G and S_H is a clique in H*

Proof: Let S be a subset of $V(G \oplus H)$. Then either $S \subseteq V(G)$, $S \subseteq V(H)$, or S intersects both $V(G)$ and $V(H)$. Assume that S is a clique in $G \oplus H$.
Case 1*:* $S \subseteq V(G)$.

By definition of the join of graphs, the induced subgraph of S in G is the same as the induced subgraph of S in $G \oplus H$. Hence, S is a clique in G and (i) is satisfied.

Case 2*:* $S \subseteq V(H)$.

A similar argument to Case 1 implies (ii).

Case 3*:* $S \cap V(G) \neq \varnothing$ and $S \cap V(G) \neq \varnothing$.

Necessarily, $S \cap V(G)$ must induce a complete graph in G and $S \cap V(H)$ must induce a complete graph in H. Take $S_G = S \cap V(G)$ and $S_H = S \cap V(H)$. Then, condition (iii) follows.

For the converse, suppose that $S \subset V(G \oplus H)$ satisfies (i); that is, S is a clique in G. This implies that $S \subseteq V(G)$. Note that, in this case, the induced subgraph of S is G is the same as the induced subgraph of S in $G \oplus H$. Hence, S induces a complete graph in $G \oplus H$. Accordingly, S is a clique in $G \oplus H$. A similar argument when S is a clique in H asserts that S is a clique in $G \oplus H$. Assume that (iii) is satisfied. By the adjacency property of the join of graphs, S induces a complete graph in $G \oplus H$. Accordingly, S is a clique in $G \oplus H$. This completes the proof of the lemma. $\square$

The following establishes our main result on the clique connected common neighborhood polynomial of the graph resulting from the join of two connected graphs.

Theorem 2.3. *Let G and H be connected graphs. Then*

$$\begin{aligned}\Omega_{cccn}(G \oplus H; x, y) &= |V(H)|\Omega_{cccn}(G; x, y) + |V(G)|\Omega_{cccn}(H; x, y) \\ &\quad + ccn(G; x, y)ccn(H; x, y).\end{aligned}$$

Proof: Let S be a clique in $V(G \oplus H)$. By Lemma 2.2 (i) and by the connectivity property of H, the entire vertex-set of H adds to the connected common neighborhood of every clique in G. This gives the first term. The second term follows a similar argument. Now, suppose that $S_G = S \cap V(G)$ and $S_H = S \cap V(H)$ are non-empty. From Lemma 2.2 (iii), since $V(H)$ is in the neighborhood of S_G, the neighborhood of S_G in G is contained in the neighborhood of S in $G \oplus H$. Similarly, the neighborhood of S_H in H is contained in the neighborhood of S in $G \oplus H$. Moreover, these neighborhoods are common to S and are connected by the connectivity property of the join of G and H. The third term follows. $\square$

3 Conclusion

In this study, we have established the clique connected common neighborhood polynomial of the graph resulting from the join of two graphs in terms of the clique common neighborhood and the clique connected common neighborhood polynomials of the graphs being considered. We conclude that even if the neighborhood of the cliques in G or H is not connected, these neighborhoods are connected common neighborhoods in their join.

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