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Geodetic Bounds in Graphs

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Abstract

Let G be a connected graph. A subset S of $V(G)$ is said to be geodetically bounded if there exist $u, v \in V(G)$ such that $S \subseteq I_G[u, v]$. The geodetic bounding number of G is the cardinality of the maximum geodetically bounded subset of $V(G)$. In this paper, we establish the properties of the geodetic bounding number of a graph as well as the geodetic bounding number of some special graphs.

1 Introduction

In the year 2000, Chartrand et al. [1] studied geodetic sets in graphs. For two vertices u and v in a graph G, a *u-v geodesic* in G is the shortest path

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joining u and v in G. The *geodetic closure* of $\{u, v\}$ in G is the set $I_G[u, v] = \{u, v\} \cup \{y : y$ lies in a u-v path in $G\}$. The *geodetic closure of a subset* S of $V(G)$ is the set $I_G[S] = \bigcup_{u,v \in S} I_G[u, v]$. A subset S of $V(G)$ is said to be a *geodetic set* if $I_G[S] = V(G)$. The minimum cardinality of a geodetic set in G is called the *geodetic number of* G and is denoted by $g(G)$.

The distance $d_G(u, v)$ between two vertices u and v in a connected graph G is the length of the shortest u-v path in G. For a vertex v of G, the eccentricity $e_G(v)$ of a vertex v in G is the distance between v and the vertex farthest from v in G. The maximum eccentricity is the *diameter* of G denoted by $diam(G)$.

A subset S of $V(G)$ is said to be *geodetically bounded* if there exists $(u, v) \in V(G) \times V(G)$ such that $S \subseteq I_G[u, v]$. In this case, the vertices u and v are geodetic bounds for S. If $V(G)$ is a geodetically bounded set, then we say that G is a *geodetically bounded graph*. The *geodetic bounding number* of G is the cardinality of the maximum geodetically bounded subset of $V(G)$. In symbol,

$$\begin{aligned}\Gamma_{gbn}(G) &= \max\{|S| : S \text{ is geodetically bounded in } G\}\\ &= \max\{|I_G[u, v]| : u, v \in V(G)\}.\end{aligned}$$

Graphs considered in this study are simple, connected, and undirected. For standard terminologies and notations, the readers may refer to [2, 3].

2 Properties and Existence

Remark 2.1. *A geodetic set may not be geodetically bounded. Moreover, a geodetically bounded set may not be a geodetic set.*

On the other hand, for $n \geq 3$, $g(K_n) = n$. Hence, the geodetic set for K_n is $V(K_n)$. Moreover, $V(K_n)$ is not geodetically bounded.

Every singleton set is geodetically bounded. Indeed, if $S = \{u\}$, then $I_G[u, u] = \{u\} = S$. In this case, if G is non-trivial and noncomplete, then S is not a geodetic set.

Every pair of vertices $\{u, v\}$ such that $uv \in E(G)$ is a geodetically bounded set. In fact, $I_G[u, v] = \{u, v\}$.

Remark 2.2. *For every $u, v \in V(G)$, $dist_G(u, v) \leq |I_G[u, v]| - 1$ and equality holds if and only if the path joining u and v in G is unique.*

The next result establishes the sharp bounds for the geodetic bounding number of graphs.

Theorem 2.3. *For a nontrivial connected graph G,*

$$\Gamma_{gbn}(G) \geq diam(G) + 1. \tag{2.1}$$

Proof. The lower bound is attained by paths. $\square$

The following can be verified easily.

Example 2.4. *For $n \geq 2$,*

1. $\Gamma_{gbn}(K_n) = 2$.
2. $\Gamma_{gbn}(P_n) = n$.

The following result establishes the properties of geodetically bounded sets and geodetic bounds of graphs.

Lemma 2.5. *A geodetically bounded set S of G whose cardinality corresponds to the geodetic bounding number of G contains its geodetic bounds.*

The result below gives the exact value for the geodetic bounding number of trees.

Theorem 2.6. *Let T_n be a tree of order $n \geq 2$. Then $\Gamma_{gbn}(T_n) = 1 + diam(T_n)$.*

Proof. By Theorem 2.3, $\Gamma_{gbn}(T_n) \geq 1 + diam(T_n)$. Note that, for every pair of vertices $\{u, v\}$, the $u - v$ geodesic in T_n is unique. Taking the maximum of any geodetic closure of pairs of vertices gives one less than the diameter of T_n. The result now follows. $\square$

The following lemma characterizes the geodetically bounded sets in the complete bipartite graph $K_{m,n}$.

Lemma 2.7. *Let m and n be natural numbers with $m, n \geq 2$ and $K_{m,n} = \overline{K_m} \oplus \overline{K_n}$. A nontrivial subset S of $V(K_{m,n})$ is geodetically bounded in $K_{m,n}$ if and only if it satisfies one of the following:*

(i) $S = S_1 \cup S_2$, *where* $S_1 \subseteq V(\overline{K_m})$, $S_2 \subseteq V(\overline{K_n})$ *and* $|S_1| = |S_2| = 1$;

(ii) $S = S_1 \cup V(\overline{K_n})$, *where* $S_1 \subseteq V(\overline{K_m})$ *and* $|S_1| = 2$; *or*

(iii) $S = V(\overline{K_m}) \cup S_2$, *where* $S_2 \subseteq V(\overline{K_n})$ *and* $|S_1| = 2$.

Proof. Let S be a geodetically bounded set. Then there exist $u, v \in V(K_{m,n})$ such that $S \subseteq I_{K_{m,n}}[u, v]$. Consider the following cases:

Case 1: $u \in V(\overline{K_m})$ and $v \in V(\overline{K_n})$. In this case, $I_{K_{m,n}}[u, v] = \{u, v\}$ since $uv \in E(K_{m,n})$. Then (i) holds.

Case 2: $u, v \in V(\overline{K_m})$. Then $I_{K_{m,n}}[u, v] = \{u, v\} \cup V(\overline{K_n})$ and (ii) is satisfied.

Case 3: $u, v \in V(\overline{K_n})$. This is similar to Case 2 and (iii) follows.

The converse is clear. □

The following establishes the geodetic bounding number of the complete bipartite graph $K_{m,n}$.

Theorem 2.8. *For* $m, n \geq 2$, $\Gamma_{gbn}(K_{m,n}) = 2 + \max\{m, n\}$.

Proof. Let S be a geodetically bounded set in $K_{m,n}$. Then it satisfies one of the conditions in Lemma 2.7. If S satisfies Lemma 2.7 (i), then $|S| = 2$. If S satisfies Lemma 2.7 (ii), then $|S| = 2 + n$. If S satisfies Lemma 2.7 (iii), then $|S| = 2 + m$. The result follows by taking the maximum of the three cases. □

Define the *parity indicator function* on $\mathbb{N}$ as the function defined by

$$i_p(n) = \begin{cases} 1 & , \text{ if } n \text{ is even} \\ 0 & , \text{ if } n \text{ is odd} \end{cases}$$

Theorem 2.9. *For* $n \geq 4$,

$$\Gamma_{gbn}(C_n) = n[i_p(n)] + \frac{n+1}{2}[1 - i_p(n)].$$

Proof. Let $C_n = [v_1, v_2, \ldots, v_n, v_1]$. Suppose n is even. Take $u = v_1$ and $v = v_{n/2}$. Then $I_{C_n}[u, v] = V(C_n)$. Hence, $\Gamma_{gbn}(C_n) = n$.

Suppose n is odd. Take $u = v_1$ and $v = v_{(n+1)/2}$. Then $I_{C_n}[u, v] = [v_1, v_2, \ldots, v_{(n+1)/2}]$. Moreover, this choice of u and v gives the maximum geodetic closure.

The result follows. □

The following result assures the existence of a geodetically bounded graph of all orders.

Theorem 2.10. *For every natural number* n, *there exists a geodetically bounded graph* G *of order* n.

Proof. Take $G = P_n$. □

The existence of graphs with prescribed order and geodetic bounding number is established in the next result.

Theorem 2.11. *Let n and k be natural numbers such that $2 \leq k < n$. Then there exists a graph G of order n with geodetic bounding number equal to k.*

Proof. For $k = 2$, take $G = K_n$. For $3 \leq k \leq n$, take $P_k = [v_1, v_2, \ldots, v_k]$ and then attach $n - k$ vertices to v_2. The resulting graph is a tree of diameter $k - 1$ and geodetic bounding number equal to k.

□

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