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On the Diophantine equation $6^x + p^y = z^2$, where p is prime

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Abstract

In this work, we study the Diophantine equation $6^x + p^y = z^2$, where p is prime and x, y, z are non-negative integers. We prove that if $p \leq 7$ and x is even, then the Diophantine equation has only four non-negative integer solutions which are $(p, x, y, z) \in \{(2, 0, 3, 3)\} \cup \{(3, 0, 1, 2), (2, 2, 6, 10), (3, 4, 6, 45)\}$. Moreover, the equation has no non-negative integer solution when it satisfies one of the following cases:
Case 1: $p \equiv 1 \pmod 5$
or
Case 2: $p \equiv 4 \pmod 5$ and y is even
or
Case 3: $p \equiv 1 \pmod 7$ and $z \not\equiv 0 \pmod 7$.

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1 Introduction

In 2012, Chotchaisthit [2] showed that all non-negative integer solutions of the Diophantine equation $4^x + p^y = z^2$, where p is prime, are of the following form $(x, p, y, z) \in \{(2, 3, 2, 5)\} \cup \{(r, 2^{r+1} + 1, 1, 2^r + 1) : r \in \mathbb{N} \cup \{0\}\} \cup \{(r, 2, 2r + 3, 3 \cdot 2^r) : r \in \mathbb{N} \cup \{0\}\}$. In 2015, Qi and Li [5] studied positive integer solutions of the Diophantine equation $8^x + p^y = z^2$, where p is an odd prime. In 2016, Cheenchan el at. [1] found that the Diophantine equation $p^x + 5^y = z^2$, where p is prime and p satisfies one of the following cases:
Case 1: $p \equiv 1 \pmod 4$
or
Case 2: $p \equiv 3 \pmod 4$ and $p \equiv 2 \pmod 5$
or
Case 3: $p \equiv 3 \pmod 4$ and $p \equiv 3 \pmod 5$,
has no non-negative integer solution.
In 2021, Tangjai and Chubthaisong [7] studied the Diophantine equation $3^x + p^y = z^2$, where p is prime and $p \equiv 2 \pmod 3$. They found that if $y = 0$, then $(p, x, y, z) = (p, 1, 0, 2)$ is the only non-negative integer solution of the equation for each prime p. If $4 \nmid y$, then $(p, x, y, z) = (2, 0, 3, 3)$ is the unique non-negative integer solution. In 2022, Pakapongpun and Chattae [4] studied non-negative integer solutions of the Diophantine equation $p^x + 7^y = z^2$, where p is prime, and proved that if $p \equiv 3 \pmod 6$, then the Diophantine equation has exactly two non-negative integer solutions $(x, y, z) \in \{(1, 0, 2), (2, 1, 4)\}$. Moreover, they found some conditions of non-existence of non-negative integer solutions. In the same year, Tadee [6] studied all non-negative integer solutions of the Diophantine equation $2^x + p^y = z^2$, where $x \neq 1$ and p is prime with $p \equiv 3 \pmod 4$. Inspired by the work mentioned earlier, we will study the Diophantine equation $6^x + p^y = z^2$, where p is prime and x, y, z are non-negative integers.

2 Main Results

We begin this section by the helpful Theorem.

Theorem 2.1 (Mihăilescu's Theorem). *[3] The Diophantine equation $a^x - b^y = 1$ has the unique solution $(a, b, x, y) = (3, 2, 2, 3)$, where a, b, x and y are integers with $min\{a, b, x, y\} > 1$.*

Lemma 2.2. *Let p be prime. Then the Diophantine equation $1 + p^y = z^2$ has only two non-negative integer solutions $(p, y, z) \in \{(2, 3, 3), (3, 1, 2)\}$.*

Proof. Let y and z be non-negative integers such that $1 + p^y = z^2$. It is easy to check that $z > 1$ and $y \geq 1$. If $y = 1$, then $p = (z-1)(z+1)$. Since p is prime, we have $z - 1 = 1$ and $z + 1 = p$. Then $z = 2$ and $p = 3$. That is $(p, y, z) = (3, 1, 2)$. If $y > 1$, then $\min\{z, p, 2, y\} > 1$. By Theorem 2.1, we have $(p, y, z) = (2, 3, 3)$. □

Theorem 2.3. *Let* p *be prime. If* $p \leq 7$ *and* x *is even, then the Diophantine equation* $6^x + p^y = z^2$ *has only four non-negative integer solutions* $(p, x, y, z) \in \{(2, 0, 3, 3), (3, 0, 1, 2), (2, 2, 6, 10), (3, 4, 6, 45)\}$.

Proof. Let x, y and z be non-negative integers such that $6^x + p^y = z^2$. Since x is even, we get $x = 2k$ for some non-negative integer k. If $k = 0$, then $(p, x, y, z) \in \{(2, 0, 3, 3), (3, 0, 1, 2)\}$, by Lemma 2.2. Now, we consider $k \geq 1$. Then $(z - 6^k)(z + 6^k) = p^y$. Since p is prime, there exists an integer u with $0 \leq u \leq y$ such that $z - 6^k = p^u$ and $z + 6^k = p^{y-u}$. Thus $y > 2u$ and so $2^{k+1} \cdot 3^k = p^u(p^{y-2u} - 1)$.

Case 1: $p = 2$. Then $u = k + 1$ and $2^{y-2u} - 3^k = 1$. If $y - 2u = 1$, then $k = 0$, a contradiction. Then $y - 2u > 1$. If $k > 1$, then $\min\{2, 3, y - 2u, k\} > 1$. This is impossible by Theorem 2.1. Thus $k = 1$ and so $(p, x, y, z) = (2, 2, 6, 10)$.

Case 2: $p = 3$. Then $u = k$ and $3^{y-2u} - 2^{k+1} = 1$. If $y - 2u = 1$, then $k = 0$, a contradiction. Then $y - 2u > 1$ and so $\min\{3, 2, y - 2u, k + 1\} > 1$. By Theorem 2.1, we have $y - 2u = 2$ and $k + 1 = 3$. Thus $k = u = 2$ and $y = 6$. It follows that $(p, x, y, z) = (3, 4, 6, 45)$.

Case 3: $p = 5$. Then $u = 0$ and $2 \cdot 6^k = 5^y - 1$. Since $y > 0$, we obtain $5^y - 1 \equiv -1 \pmod 5$. Then $2 \cdot 6^k \equiv -1 \pmod 5$. This is impossible since $2 \cdot 6^k \equiv 2 \pmod 5$.

Case 4: $p = 7$. Then $u = 0$ and $2 \cdot 6^k = 7^y - 1$. Since $y > 0$, we get $7^y - 1 \equiv -1 \pmod 7$. Then $2 \cdot 6^k \equiv -1 \pmod 7$. This is impossible since $2 \cdot 6^k \equiv 2, 5 \pmod 7$. □

Theorem 2.4. *Let* p *be prime. If* $p \equiv 1 \pmod 5$*, then the Diophantine equation* $6^x + p^y = z^2$ *has no non-negative integer solution.*

Proof. Assume that there exists non-negative integers x, y and z such that $6^x + p^y = z^2$. Since $p \equiv 1 \pmod 5$, we have $6^x + p^y \equiv 2 \pmod 5$. Then $z^2 \equiv 2 \pmod 5$. This is impossible since $z^2 \equiv 0, 1, 4 \pmod 5$. □

Theorem 2.5. *Let p be prime. If $p \equiv 4 \pmod 5$ and y is even, then the Diophantine equation $6^x + p^y = z^2$ has no non-negative integer solution.*

Proof. Assume that there exists non-negative integers x, y and z such that $6^x + p^y = z^2$. Since $p \equiv 4 \pmod 5$ and y is even, we have $p^y \equiv 1 \pmod 5$. Then $6^x + p^y \equiv 2 \pmod 5$ and so $z^2 \equiv 2 \pmod 5$. This is impossible since $z^2 \equiv 0, 1, 4 \pmod 5$. □

Theorem 2.6. *Let p be prime. If $p \equiv 1 \pmod 7$ and $z \not\equiv 0 \pmod 7$, then the Diophantine equation $6^x + p^y = z^2$ has no non-negative integer solution.*

Proof. Assume that there exists non-negative integers x, y and z such that $6^x + p^y = z^2$. Since $z \not\equiv 0 \pmod 7$, we get $z^2 \not\equiv 0 \pmod 7$. Since $p \equiv 1 \pmod 7$, we have $6^x + p^y \equiv (-1)^x + 1 \pmod 7$. Thus $0 \not\equiv (-1)^x + 1 \pmod 7$ and so x is even. There exists a non-negative integer h such that $x = 2h$. Then $(z - 6^h)(z + 6^h) = p^y$. Since p is prime, there exists an integer v with $0 \leq v \leq y$ such that $z - 6^h = p^v$ and $z + 6^h = p^{y-v}$. Thus $y > 2v$ and $2 \cdot 6^h = p^v(p^{y-2v} - 1)$. Since p is prime and $p \equiv 1 \pmod 7$, we get $p \notin \{2, 3\}$. Then $v = 0$ and so $2 \cdot 6^h = p^y - 1$. Since $p^y - 1 \equiv 0 \pmod 7$, we get $2 \cdot 6^h \equiv 0 \pmod 7$. This is impossible since $2 \cdot 6^h \equiv 2, 5 \pmod 7$. □

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References

[1] I. Cheenchan, S. Phona, J. Ponggan, S. Tanakan, S. Boonthiem, On the Diophantine equation $p^x + 5^y = z^2$, *SNRU Journal of Science and Technology*, **8**, no. 1, (2016), 146–148.

[2] S. Chotchaisthit, On the Diophantine equation $4^x + p^y = z^2$ where p is a prime number, *Amer. J. Math. Sci.*, **1**, no. 1, (2012), 191–193.

[3] P. Mihăilescu, Primary cyclotomic units and a proof of Catalan's conjecture, *J. Reine Angew. Math.*, **572**, (2004), 167–195.

[4] A. Pakapongpun, B. Chattae, On the Diophantine equation $p^x + 7^y = z^2$, where p is prime and x, y, z are non-negative integers, *Int. J. Math. Comput. Sci.*, **17**, no. 4, (2022), 1535–1540.

[5] L. Qi, X. Li, The Diophantine equation $8^x + p^y = z^2$, *The Scientific World Journal*, (2015), Article ID 306590, 3 pages.

[6] S. Tadee, On the Diophantine equation $2^x + p^y = z^2$, where $x \neq 1$ and $p \equiv 3 \pmod 4$, *Mathematical Journal by The Mathematical Association of Thailand Under The Patronage of His Majesty The King*, **67**, no. 707, (2022), 13–19.

[7] W. Tangjai, C. Chubthaisong, On the Diophantine equation $3^x + p^y = z^2$, where $p \equiv 2 \pmod 3$, *WSEAS Transactions on Mathematics*, **20**, (2021), 283–287.