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Modules whose δ-small epimorphisms are isomorphisms

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Abstract

An R-module M is called δ-weakly Hopfian if any δ-small surjective endomorphism of M is an automorphism. In this paper we explore various properties of δ-weakly Hopfian modules, shedding light on their distinct characteristics. Additionally, we examine the δ-weakly Hopficity of modules over polynomial and truncated polynomial rings.

1 Introduction

Throughout this paper all rings have identity and all modules are unital right modules. The concept of Hopfian modules was introduced by Hiremath [7]. An R-module M is Hopfian if every surjective endomorphism of M is an automorphism. Moreover, Co-Hopfian modules, introduced by Varadarajan, are modules where every injective endomorphism is an automorphism. Furthermore, generalized Hopfian modules, introduced and studied by Ghorbani and Haghany [6], are modules where any surjective endomorphism has a small kernel. Weakly Hopfian modules, a proper generalization of Hopfian modules, were defined in [8], a right R-module M is weakly Hopfian if any

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small surjective endomorphism of M is an automorphism. It was established that a right R-module M is Hopfian if and only if it is both generalized Hopfian and weakly Hopfian. These concepts, along with other generalizations, have been extensively studied by various authors ([2], [3], [4], [5]).

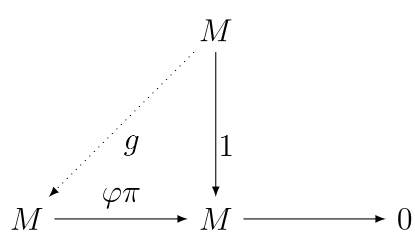
In [1], the authors introduced δ-weakly Hopfian modules, where a right R-module M is δ-weakly Hopfian if every δ-small surjective endomorphism of M is an automorphism.

2 Modules whose δ-small epimorphisms are isomorphism

Definition 2.1. *[1] Let M be an R-module. We say that M is δ-weakly Hopfian if every δ-small surjective endomorphism of M is an automorphism.*

Proposition 2.2. *Every quasi-projective uniform right R-module is δ-weakly Hopfian.*

Proof.
Let M be a quasi-projective uniform R-module. Suppose $M \cong M/K$ for some $K \ll_\delta M$. Let $\varphi : M/K \to M$ be an isomorphism. The map $\varphi\pi : M \to M$, where $\pi : M \to M/K$ is canonical epimorphism has kernel K; i.e., $Ker(\varphi\pi) = K$. Since M is quasi-projective, there is a $g : M \to M$ which makes the following diagram commutative.



Then $M = Ker(\varphi\pi) \oplus Im(g)$. Since M is uniform, $Im(g) \leq^e M$. Hence $M/Im(g)$ is singular. Since $Ker(\varphi\pi) = K \ll_\delta M$, we must have $K = 0$ and so M is δ-weakly Hopfian by [1, Lemma 2.3].

Corollary 2.3. *Let M be a quasi-projective right R-module such that every nonzero submodule of M is indecomposable. Then M is δ-weakly Hopfian.*

Proof.
An R-module M is uniform if and only if every nonzero submodule of M is indecomposable. Then M is δ-weakly Hopfian by Proposition 2.2.

Theorem 2.4. *Let M be an R-module. If $M[x]$ is a δ-weakly Hopfian $R[x]$-module, then M is a δ-weakly Hopfian R-module.*

Proof.
Let $f : M \to M$ be any δ-small epimorphism in R-module. Then $\alpha : M[x] \to M[x]$ defined by $\alpha(\sum_{i=0}^{n} a_i x^i) = \sum_{i=0}^{n} f(a_i)x^i$ is a surjective endomorphism in an $R[x]$-module and $Ker\alpha = Ker(f)[x]$. Assume that $Ker\alpha + N = M[x]$ for all submodule N of $M[x]$ with a singular factor over $R[x]$. Therefore, $Kerf + N' = M$ where N' is the submodule of M which is generated by the constant polynomials of N. We show that M/N' is a singular R-module. Let $m \in M$. Since $M[x]/N$ is singular, there exists $I[x] \leq^e R[x]$ (essential), such that $mI[x] \subseteq N$. Therefore, $I \leq^e R$ and $mI \subseteq N'$. Then M/N' is singular. Since $Kerf$ is δ-small in M, $M = N'$. Hence $M[x] = N$ and $Ker\alpha = Kerf[x]$ is δ-small in $M[x]$. Since $M[x]$ is δ-weakly Hopfian $R[x]$-module, α is an automorphism in $M[x]$. Consequently, f is an automorphism in M, and finally M is δ-weakly Hopfian.

Corollary 2.5. *Let M be an R-module. If $M[x_1, ..., x_k]$ is δ-weakly Hopfian $R[x_1, ..., x_k]$-module, then M is δ-weakly Hopfian R-module.*

Proof.
Use induction and the $R[x_1, ..., x_{k-1}][x_k]$-module isomorphism
$M[x_1, ..., x_{k-1}][x_k] \simeq M[x_1, ..., x_k]$, and ring isomorphism
$R[x_1, ..., x_{k-1}][x_k] \simeq R[x_1, ..., x_k]$.

Theorem 2.6. *Let M be an R-module. If $M[x]/(x^{n+1})$ is δ-weakly Hopfian $R[x]/(x^{n+1})$-module, then M is δ-weakly Hopfian R-module.*

Proof.
The proof is similar to that of theorem 2.4.

Corollary 2.7. *Let M be an R-module. If $M[x_1, ..., x_k]/(x_1^{n_1+1}, ..., x_k^{n_k+1})$ is δ-weakly Hopfian $R[x_1, ..., x_k]/(x_1^{n_1+1}, ..., x_k^{n_k+1})$-module, then M is δ-weakly Hopfian R-module.*

Proof.
Use induction and the
$(R[x_1, ..., x_{k-1}]/(x_1^{n_1+1}, ..., x_{k-1}^{n_{k-1}+1}))[x_k]/(x_k^{n_k+1})$-module isomorphism
$(M[x_1, ..., x_{k-1}]/(x_1^{n_1+1}, ..., x_{k-1}^{n_{k-1}+1}))[x_k]/(x_k^{n_k+1}) \simeq M[x_1, ..., x_k]/(x_1^{n_1+1}, ..., x_k^{n_k+1})$
and ring isomorphism
$(R[x_1, ..., x_{k-1}]/(x_1^{n_1+1}, ..., x_{k-1}^{n_{k-1}+1}))[x_k]/(x_k^{n_k+1}) \simeq R[x_1, ..., x_k]/(x_1^{n_1+1}, ..., x_k^{n_k+1})$.

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