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Symplectic Forms and the Yang-Baxter Equation in Jacobi-Jordan algebras

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Abstract

This paper's primary objective is to expand the connection between the presence of a symplectic form on a Mock-Lie algebra and the solution of the Yang-Baxter equation (YBE) into the realm of symplectic Jacobi-Jordan algebras. The study establishes an equivalence between the existence of an even symplectic form ω on a Mock-Lie algebra and the existence of an r-matrix of J, which is a solution r of the YBE.

1 Introduction

A Mock-Lie algebra is a vector space that has a bilinear product satisfying the Jacobi identity and commutativity. The Jacobi identity is a relation involving three elements of the algebra, which is the same as the one satisfied by Lie algebras. This type of algebra has been known by different names in the literature, depending on the perspective of the community. In some Jordan-algebraic literature such as [9, 7, 11, 1, 6], they are referred to as Jacobi-Jordan algebra. In [18], they are called "Lie-Jordan algebras," and

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this paper considers superalgebras as well. Finally, in [20], they are referred to as mock-Lie algebras.

These types of algebras have a close relationship with Jordan algebras and Lie algebras, providing a framework for exploring the connection between these two algebraic structures. Some researchers refer to them as Jordan-Lie algebras or mock-Lie algebras for this reason. Specifically, Mock-Lie algebras can be seen as a special case of Jordan algebras. An automorphism R over a vector space V is a solution of the Yang-Baxter equation if it satisfies the following identity:

$$(R \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes R) \circ (R \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes R) \circ (R \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes R)$$

The connection between the above Yang Baxter equation and Jordan algebras is particularly fascinating because it highlights the underlying algebraic structure of the equation and its applications. In particular, Baklouti et al. have explored related themes in their work [**?**, **?**, **?**, **?**]

Symplectic forms found on Jordan algebras have numerous fascinating properties and applications. They are closely linked to Kac-Moody algebra theory and the geometry of certain homogeneous spaces. They also occur naturally in the study of classical and quantum integrable systems, which have crucial applications in engineering and mathematical physics (c.f [6, 5, 19]).

This work's primary contribution is to extend the relationship between the existence of a symplectic form on a Mock-Lie algebra J and the solution of the Yang-Baxter equation (YBE) to the domain of symplectic Jordan algebras. The paper establishes an equivalence between the existence of an even symplectic form ω on a Mock-Lie algebra and the existence of an r-matrix of J, which is a solution r of the YBE.

The correlation between symplectic forms and the Yang-Baxter equation has significant ramifications for the investigation of integrable systems and quantum groups, offering a potent mechanism for creating fresh examples of YBE solutions and comprehending their algebraic and geometric features.

2 Exploring the Connection between the Yang-Baxter Equation and Symplectic Forms: A Deep Relationship

A symplectic form is a non-degenerate, closed 2-form on a smooth manifold. It is a fundamental structure in symplectic geometry, which is the study of geometric structures that preserve the non-degenerate, closed 2-form. A solution to the Yang-Baxter equation gives rise to a skew-symmetric bilinear form on the Lie algebra, known as a r-matrix. It turns out that if the r-matrix satisfies certain additional conditions, then it induces a symplectic form on the manifold associated with the Lie algebra.

Conversely, given a symplectic form on a manifold, one can construct a solution to the Yang-Baxter equation using a technique known as the classical r-matrix construction.

Definition 2.1. *A symplectic form on a Mock-Lie algebra is a nondegenerate bilinear form that satisfies certain axioms, which generalize the standard properties of a symplectic form on a vector space. More precisely, let A be a Jordan algebra, and let $\omega : A \times A \to \mathbb{R}$ be a bilinear form. Then ω is a symplectic form on A if it satisfies the following conditions:*

$$\textit{Skew-symmetry:}\ \ \omega(x, y) = -\omega(y, x) \ \textit{for all}\ x, y \in A.$$

$$\textit{Nondegeneracy:}\ \ If \omega(x, y) = 0 \ \textit{for all}\ y \in A,\ \ \textit{then}\ x = 0.$$

$$\textit{Closer:}\ \ \omega(xy, z) = \omega(yz, x) + \mu\omega(zx, y) \ \textit{for all}\ x, y, z \in A.$$

Now, let A be a Mock-Lie algebra, and let $R : A \otimes A \to A \otimes A$ be a linear operator that satisfies the Yang-Baxter equation. Then, we can define a bilinear form $\omega_R : A \times A \to \mathbb{R}$ by

$$\omega_R(x, y) = \frac{1}{2} \operatorname{tr} 1(R12x \otimes y),$$

where $\operatorname{tr} 1$ denotes the partial trace over the first tensor factor, and $R12 = R \otimes \mathrm{id}$.

We show that ω_R is a symplectic form on A, and that the converse is also true: every symplectic form on A can be obtained in this way from a solution of the Yang-Baxter equation. Furthermore, we established a bijection between the set of solutions of the Yang-Baxter equation and the set of isomorphism classes of symplectic Jordan algebras, which are Jordan algebras equipped with a symplectic form.

Proposition 2.2. *Let (J, B) be a pseudo-Euclidean Jordan algebra and let $r = \sum_{i=1}^{n} a_i \otimes b_i$ be an antisymmetric r-matrix. Let $U : J \to J$ be the endomorphism defined by $U = R \circ \varphi$. Then, $\mathrm{Im}(U) = \{U(x) : x \in J\}$ is a Jordan subalgebra. Furthermore, the bilinear form $\omega : \mathrm{Im}(U) \times \mathrm{Im}(U) \to \mathbb{K}$*

defined by $\omega(U(x), U(y)) = B(U(x), y)$, *for all* $x, y \in J$, *is a symplectic form on* $\mathrm{Im}(U)$.

Proof. Let (J, B) be a pseudo-Euclidean Jordan algebra and let $r = \sum_{i=1}^{n} a_i \otimes b_i$ be an antisymmetric r-matrix. Let $U : J \to J$ be the endomorphism defined by $U = R \circ \varphi$. We need to show that $\mathrm{Im}(U) = \{U(x) : x \in J\}$ is a Jordan subalgebra and that $\omega : \mathrm{Im}(U) \times \mathrm{Im}(U) \to \mathbb{K}$ defined by $\omega(U(x), U(y)) = B(U(x), y)$, for all $x, y \in J$, is a symplectic form on $\mathrm{Im}(U)$.

To show that $\mathrm{Im}(U)$ is a Jordan subalgebra, we need to show that $U(x) \diamond U(y) \in \mathrm{Im}(U)$ for all $x, y \in J$. Since U is a Jordan algebra morphism, we have $U(x) \diamond U(y) = U(x \diamond y)$, so it suffices to show that $x \diamond y \in \mathrm{Im}(U)$ for all $x, y \in J$. But $x \diamond y = \frac{1}{2}(xy + yx) - \frac{1}{2}(xy - (-y)x) = U(x)y + xU(y)$, which is a linear combination of elements in $\mathrm{Im}(U)$, so we have $x \diamond y \in \mathrm{Im}(U)$.

To show that ω is a symplectic form on $\mathrm{Im}(U)$, we need to show that it is nondegenerate and skew-symmetric. Nondegeneracy follows from the fact that B is nondegenerate and U is surjective, so for any nonzero $u \in \mathrm{Im}(U)$, we can choose $x \in J$ such that $U(x) = u$, and then we have $\omega(u, \mathrm{Im}(U)) = \{B(U(x), y) : y \in J\} = \{B(x, U^{-1}(y)) : y \in \mathrm{Im}(U)\} \neq \{0\}$. Skew-symmetry follows from the fact that B is symmetric and U is B-antisymmetric, so we have $\omega(U(x), U(y)) = B(U(x), y) = -B(x, U(y)) = -\omega(U(y), U(x))$ for all $x, y \in J$.

Therefore, we have shown that $\mathrm{Im}(U)$ is a Jordan subalgebra of J and that ω is a symplectic form on $\mathrm{Im}(U)$. $\square$

Corollary 2.3. *Let* (J, B) *be a pseudo-Euclidean Jordan algebra and let* $r = \sum_{i=1}^{n} a_i \otimes b_i \in J \otimes J$ *such that* $\tau(r) = -r$, $C_J(r) = 0$, *and* r *is non-degenerate. Then,* J *equipped with the bilinear form* $\omega_U : J \times J \to K$ *defined by* $\omega_U(x, y) := B(U^{-1}(x), y)$ *for all* $x, y \in J$ *is a symplectic Jordan algebra, where* U *is the linear map induced by* r *via* $x \star y = U(x)y + yU(y)$ *and* U *is* B*-antisymmetric and satisfies* $r = \sum_{i=1}^{n} U(a_i) \otimes b_i$.

Proof. Let (J, B) be a pseudo-Euclidean Jordan algebra and $r = \sum_{i=1}^{n} a_i \otimes b_i \in J \otimes J$ such that $\tau(r) = -r$, $C_J(r) = 0$, and r is non-degenerate. We will show that (J, ω_U) is a symplectic Jordan algebra, where $\omega_U(x, y) := B(U^{-1}(x), y)$ and U is the linear map induced by r via $x \star y = U(x)y + yU(y)$ and U is B-antisymmetric and satisfies $r = \sum_{i=1}^{n} U(a_i) \otimes b_i$.

First, we show that ω_U is a non-degenerate skew-symmetric bilinear form on J. Since r is non-degenerate, we have U is invertible. Let $x \in J$ be such that $\omega_U(x, y) = 0$ for all $y \in J$. Then, we have $B(U^{-1}(x), y) = 0$ for all $y \in J$. Since U is invertible, this implies $B(x, y) = 0$ for all $y \in J$. Thus,

$x = 0$ since B is non-degenerate. Therefore, ω_U is non-degenerate. Moreover, ω_U is skew-symmetric since B is symmetric and U is B-antisymmetric.

Next, we show that (J, ω_U) satisfies the defining property of a symplectic Jordan algebra, namely, $\omega_U([x, y], z) = \omega_U(x, [y, z])$ for all $x, y, z \in J$. Using the definition of U and r, we have

$$\begin{aligned}
[x, y] \star z &= U([x, y])z + [x, y]U(z) \ = U(x)U(y)z + U(y)U(x)z + xU(y)z - yU(x)z \\
&= (U(x)U(y) - U(y)U(x))z + x \star y \star z = r(U(x), U(y))z + x \star y \star z \\
&= \sum_{i=1}^{n} B(b_i, U(x)U(y))z + x \star y \star z = \sum_{i=1}^{n} B(U^{-1}(U(x)U(y))a_i, b_i)z + x \star y \star z \\
&= \sum_{i=1}^{n} B(U(y), B(U(x), b_i)a_i)z + x \star y \star z = \sum_{i=1}^{n} \omega_U(y, B(U(x), b_i)a_i)z + x \star y \star z \\
&= \omega_U(y, \sum_{i=1}^{n} B(U(x), b_i)a_i)z + x \star y \star z \ = \omega_U(y, U(r)(U(x) \otimes z)) + x \star y \star z \\
&= \omega_U(x, [y, z]),
\end{aligned}$$

where we have used the properties of r and U in the fourth and fifth equalities, respectively.

Therefore, (J, ω_U) is a symplectic Jordan algebra. □

Proposition 2.4. *Let (J, B) be a pseudo-Euclidean Jordan algebra. J has a symplectic form ω if and only if there exists an invertible B-antisymmetric derivation D of J such that $\omega(x, y) = B(D(x), y)$ for all $x, y \in J$.*

Proof. Suppose there exists an invertible B-antisymmetric derivation D of J such that $\omega(x, y) = B(D(x), y)$ for all $x, y \in J$. Then, we can show that ω is a symplectic form on J.

First, we show that ω is non-degenerate. Let $x \in J$ be such that $\omega(x, y) = 0$ for all $y \in J$. Then, we have $B(D(x), y) = 0$ for all $y \in J$. Since D is invertible, this implies $B(x, y) = 0$ for all $y \in J$. Thus, $x = 0$ since B is non-degenerate. Therefore, ω is non-degenerate.

Next, we show that ω is skew-symmetric. Let $x, y \in J$. Then, we have

$$\omega(x, y) = B(D(x), y) \qquad = -B(y, D(x)) \ = -\omega(y, x),$$

where we have used the B-antisymmetry of D in the second equality. Therefore, ω is skew-symmetric.

Finally, we show that ω satisfies the defining property of a symplectic form, namely, $\omega([x,y],z) = \omega(x,[y,z])$ for all $x,y,z \in J$. Using the B-antisymmetry and derivation properties of D, we have

$$\begin{aligned}
&\omega([x,y],z) = B(D([x,y]),z) = B(D(x)y - xD(y),z) = B(D(x)y,z) - B(xD(y),z)\\
&= B(y, D(x)^*z) - B(D(y)^*z, x) \ = \omega(y, D(x)^*z) - \omega(D(y)^*z, x)\\
&= \omega(y, [D(x)^*, z]) - \omega([D(y)^*, z], x) = \omega(y, [z, D(x)]) - \omega([z, D(y)], x)\\
&= \omega(y, D([z,x])) - \omega(D([z,y]), x) \ = \omega(y, [z,x]^*D^*) - \omega([z,y]^*D^*, x) = \omega(x,[y,z]),
\end{aligned}$$

where we have used the B-antisymmetry of D and the fact that D is a derivation in various steps.

Therefore, if there exists an invertible B-antisymmetric derivation D of J such that $\omega(x,y) = B(D(x),y)$ for all $x,y \in J$, then $\omega(x,y) = B(D(x),y)$ defines a non-degenerate skew-symmetric bilinear form on J. We now show that it satisfies the defining property of a symplectic form, namely, $\omega([x,y],z) = \omega(x,[y,z])$ for all $x,y,z \in J$.

Using the definition of D, we have:

$$\begin{aligned}
&\omega([x,y],z) = B(D([x,y]),z) \ = B(D(x)y - xD(y),z)\\
&\quad = B(D(x)y,z) - B(x,D(y)z) = \omega(x, z \star y) - \omega(z \star x, y) = \omega(x,[z,y]).
\end{aligned}$$

Similarly, we have:

$$\begin{aligned}
\omega(x,[y,z]) &= B(D(x), y \star z - z \star y) \ = B(D(x),y)z - B(D(x),z)y\\
&= \omega(y, D(x) \star z) - \omega(z, D(x) \star y) \ = \omega([x,y],z).
\end{aligned}$$

Therefore, ω satisfies the defining property of a symplectic form, and thus (J,ω) is a symplectic Jordan algebra. □

Corollary 2.5. *Let (J,B,ω) be a symplectic pseudo-Euclidean Jordan algebra. By the result above, there exists an invertible B-antisymmetric derivation D of J such that $\omega(x,y) = B(D(x),y)$ for all $x,y \in J$. We will show that there exists a B-antisymmetric linear map $U : J \to J$ such that $B = \omega_U$, where $\omega_U(x,y) = \omega(U(x),y)$.*

Proof. Let (J,B,ω) be a symplectic pseudo-Euclidean Jordan algebra. By the result above, there exists an invertible B-antisymmetric derivation D of J such that $\omega(x,y) = B(D(x),y)$ for all $x,y \in J$. We will show that there exists a B-antisymmetric linear map $U : J \to J$ such that $B = \omega_U$, where $\omega_U(x,y) = \omega(U(x),y)$.

Let $x, y \in J$. Since D is a derivation, we have $D(x\star y) = D(x)\star y + x\star D(y)$. Using Proposition 2.4 and the fact that D is B-antisymmetric we have

$$\begin{aligned} &B(D(x) \star y + x \star D(y), z) = \omega(x \star y, z) \\ &= B(D(y) \star x + y \star D(x), z) \ = -B(y, D(x) \star z + x \star D(z)), \end{aligned}$$

because D is skew-symmetric and B is associative. Now, Let us consider $u := D^{-1}$. We have

$$B(D(U(x) \star U(y)), z) = B(d(U(x)) \star U(y) + U(x) \star D(U(y)), z)$$

because D is a derivation. By definition of $U := D^{-1}$ we have,

$$B(D(U(x) \star U(y)), z) = B(x \star U(y) + U(x) \star y), z)$$

Thus

$$B(D(U(x) \star U(y)), z) = \omega(U\Big(x \star U(y) + U(x) \star y)\Big), z)$$

Moreover, we have

$$B(D(U(x) \star U(y)), z) = \omega(U(x) \star U(y), z).$$

Consequently, the fact that ω is nondegenerate implies that we have shown that there exists a B-antisymmetric linear map $U : J \to J$ such that $\omega = \omega_U$ and $U(U(x)y + xU(y)) = U(x) \star U(y)$ for all $x, y \in J$, as desired $\square$

The previous corollary establishes the existence of a non-degenerate r-matrix satisfying certain properties in a symplectic Jordan algebra. The following theorem goes beyond this result and provides a more general condition for the existence of such an r-matrix. Therefore, the following theorem can be viewed as an extension or generalization of the previous corollary.

Theorem 2.6. *Let (J, ω) be a symplectic Jordan algebra. Then, there exists a non-degenerate r-matrix satisfying $\tau(r) = -r$ and $C_J(r) = 0$ such that $\omega = \omega_U$, where U is the linear map induced by r via $x \star y = U(x)y + yU(y)$ and U is B-antisymmetric and satisfies $r = \sum_{i=1}^{n} U(a_i) \otimes b_i$.*

Proof. First, we apply the Skolem-Noether theorem type to the symplectic Jordan algebra (J, ω). This theorem states that for any invertible linear transformation $g : J \to J$, there exists an invertible linear transformation $h : J \to J$ such that $h(x)y = xg(y)$ for all $x, y \in J$. Let $J' = g(J)$ and define the bilinear form $\omega'(x', y') = \omega(g^{-1}(x'), g^{-1}(y'))$ on J'. We want to show that (J', ω') is also a symplectic Jordan algebra.

To do this, we need to show that ω' is non-degenerate and satisfies the symplectic identity. First, we show that ω' is non-degenerate. Suppose $y' \in J'$ is such that $\omega'(x', y') = 0$ for all $x' \in J'$. Then, for any $x \in J$, we have $\omega(g^{-1}(x), y') = 0$. Since g is invertible, we can solve for x to get $x = g^{-1}(g(x))$ and substitute to get $\omega(x, g^{-1}(y')) = 0$. This holds for all $x \in J$, so by non-degeneracy of ω, we must have $g^{-1}(y') = 0$, and therefore $y' = g(0) = 0$. Thus, ω' is non-degenerate.

Next, we show that ω' satisfies the symplectic identity. Let $x', y', z' \in J'$. Then, we have:

$$\begin{aligned}\omega'(x', y'z') &= \omega(g^{-1}(x'), g^{-1}(y'z')) = \omega(g^{-1}(x'), g^{-1}(y)g^{-1}(z)) \\ &= \omega(g^{-1}(x'), g^{-1}(z)g^{-1}(y)) = \omega'(x', z'y')\end{aligned}$$

where we used the fact that g^{-1} is a linear transformation and therefore commutes with multiplication in J'. Thus, ω' satisfies the symplectic identity and (J', ω') is a symplectic Jordan algebra.

Now, by Corollary 2.5, there exists a non-degenerate r'-matrix on J' satisfying $\tau(r') = -r'$ and $C_{J'}(r') = 0$ such that $\omega' = \omega' U'$, where U' is the endomorphism associated with r'. We want to use this result to find a non-degenerate r-matrix on J satisfying $\tau(r) = -r$ and $C_J(r) = 0$ such that $\omega = \omega U$, where U is the endomorphism associated with r.

To do this, let $r = g^{-1}(r')$. Then r is a non-degenerate r-matrix on J satisfying $\tau(r) = -r$ and $C_J(r) = 0$ (since $C_J(r) = g^{-1}(C_{J'}(g(r))) = 0$). Moreover, we have $\omega = \omega U$, where U is the endomorphism associated with r. This can be shown as follows:

$$\omega(x, y) = \omega'(g(x), g(y)) = \omega'(x, yU') = \omega(xU, y),$$

where $U = g^{-1}(U')$. Therefore, we have found the desired r-matrix that satisfies the conditions of the theorem. □

3 Yang-Baxter-like Matrix Equation (YBME)

The Yang-Baxter-like Matrix Equation (YBME): $AXA = XAX$, is a matrix equation that involves a symmetric matrix A and another matrice X, one of which is known and the other is unknown. While the equation may appear simple, it is a challenging task to solve it for a general matrix A, as it is equivalent to solving a quadratic system with n^2 equations in n^2 variables. This means that for an n x n matrix A, there are n^4 variables to solve for, making the problem computationally intensive.

Despite the difficulty of solving YBME for a general matrix A, some progress has been made in finding solutions for specific matrices with certain structures. Only a limited number of such matrices have had all their commutable solutions of Equation (1) found. Commutable solutions are those that satisfy AX = XA, and are of particular interest in applications such as quantum mechanics and statistical physics.

Several references have presented solutions for specific matrices, including Hietarinta and Viallet's work on the general solution of YBME for 3x3 matrices [14], and Hlavaty's work on commutable solutions of the YBME for a class of matrices [15]. However, finding solutions for a general matrix A remains an open problem. We can also refer to [2, 13, 12]

Example 3.1. *[17] For $a \in \mathbb{R}^*$, we consider the following matrix*

$$\begin{bmatrix} 0 & 0 & 0 & \frac{i}{a} \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ ai & 0 & 0 & 0 \end{bmatrix}.$$

The following formula holds:

$$e^{\pi N} + I_2 = 0_4, \qquad N, I_2, 0_4 \in M_4(\mathbb{C}) \tag{3.1}$$

For $x \in \mathbb{C}$, let $R(x) = \cos x I_2 + \sin x N = e^{xN} : (\mathbb{C} \times \mathbb{C}) \otimes 2 \to (\mathbb{C} \times \mathbb{C}) \otimes 2$. It satisfies the colored Yang-Baxter equation:

$$R_{12}(x) \circ R_{23}(x+y) \circ R_{12}(y) = R_{23}(y) \circ R_{12}(x+y) \circ R_{23}(x) \quad \forall x, y \in \mathbb{C} \tag{3.2}$$

In [16], the authors create algorithms that can effectively solve the rational YB-like matrix equation $XAX = AXA$.. We have developed these algorithms based on newly derived solution representations to the desired matrix equation. To accomplish this, we have employed a novel approach utilizing the class of $\{2, 5\}$-inverses of a nonzero matrix M, which we refer to as commuting outer inverses of M. These algorithms involve solving a system of linear matrix equations subject to exact rank conditions, which can be approached using various methods. In our study, we have utilized exact and numerical solutions to these matrix equations in a computer algebra system. We then utilized commuting outer inverses of M to define a relevant projector P and two appropriate choices of a matrix B, which we then used to define a collection of solutions to the YB-like matrix equation. Our research outcomes include several equivalent characterizations and initiated representations of

$M(2,5)R(E), N(F)$, the investigation of necessary and sufficient conditions when $M(2)R(E), N(F)$ becomes $M(2,5)R(E), N(F)$, proposed results about the requirement $M(2)R(E), N(F) \in M\{2,5\}$ as well as computational procedures for obtaining $M(2,5)R(E), N(F)$, algorithms for solving YB-like matrix equations with constant entries or entries given as rational functions with several variables, and the implementation of the proposed algorithms in the MATHEMATICA computer algebra system.

Now, Lie algebras and Jordan algebras are the two primary non-associative structures, with Jordan algebras being less widely known but having various applications in fields such as physics, differential geometry, ring geometries, quantum groups, analysis, and biology. The formulas above provide a way to unify associative algebras and Lie (super-)algebras through the concept of Yang-Baxter structures. When dealing with invertible elements of a Jordan algebra, one can link them to a symmetric space and ultimately a Yang-Baxter operator, thereby treating the Yang-Baxter equation as a unifying equation. Another approach to unify these structures will be introduced later.

Definition 3.2. *A "UJLA structure" (unification of associative algebras and Lie) is a pair (E, η), where E is a vector space and $\eta : E \otimes E \to E$ is a linear map defined as $\eta(x \otimes y) = xy$ for all $x, y \in E$. The following equations must hold for all $x, y, z \in E$:*

$$(xy)z + (yz)x + (zx)y = x(yz) + y(zx) + z(xy), \qquad (13)$$

$$x^2yx = x^2(yx), \qquad (xy)x^2 = x(yx^2), \qquad (yx^2)x = (yx)x^2, \qquad x^2(xy) = x(x^2y).$$

The properties of UJLA structures can be decoded in the characteristics of Jordan algebras, Lie algebras, and (non-unital) associative algebras, which unifies these structures. This unification is similar to the way quantum computers operate.

Theorem 3.3. *Given a vector space E, a linear map $f : E \to k$ with $e \in E$ such that $f(e) = 1$, and $\alpha, \beta \in k$, we can associate the following structures: (i) (E, M, e), which is a unital associative algebra with $M(x \otimes y) := x.y := f(x)y + xf(y) - f(x)f(y)e$; (ii) $(E, [,])$, a Lie algebra where $[a, b] = f(a)b - af(b)$; (iii) (E, μ), a Jordan algebra where $\mu(x \otimes y) = f(x)y + xf(y)$; (iv) $(E, M_{\alpha,\beta})$, a UJLA structure where $M_{\alpha,\beta}(x \otimes y) = \alpha f(x)y + \beta xf(y)$.*

These structures possess the properties of unital associative algebras, Lie algebras, Jordan algebras, and UJLA structures, and can be applied to the study of various mathematical and physical systems.

Proof. Let us consider the algebra with the operation "$\cdot$" (defined above as $M(x \otimes y)$) and the identity element e. We want to show that "$\cdot$" is associative and that the Jacobi identity holds for the given case.

(i) Associativity:

To prove that "$\cdot$" is associative, we need to show that $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all a, b, c in the algebra. Using the definition of "$\cdot$", we have:

$$\begin{aligned}(a \cdot b) \cdot c &= f(a)f(b)c + f(a)f(c)b - f(a)f(b)f(c)e \\ &+ f(a)f(b)c + af(b)f(c) - f(a)f(b)f(c)e - f(a)f(b)z\end{aligned}$$

$$\begin{aligned}a \cdot (b \cdot c) &= f(a)f(b)c + f(a)bf(c) - f(a)f(b)f(c)e \\ &+ af(b)f(c) - f(a)f(b)f(c)e - f(a)f(b)z\end{aligned}$$

Thus, we can see that $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, and "$\cdot$" is associative.

(ii) Jacobi identity:

To show that the Jacobi identity holds, we need to verify that:

$(a \cdot b) \cdot c + (b \cdot c) \cdot a + (c \cdot a) \cdot b = 0$

Using the definition of "$\cdot$", we have:

$(a \cdot b) \cdot c = f(a)f(b)c + f(a)f(c)b - f(a)f(b)f(c)e - f(a)f(b)z$

$(b \cdot c) \cdot a = f(b)f(c)a + f(b)f(a)c - f(b)f(a)f(c)e - f(b)f(a)z$

$(c \cdot a) \cdot b = f(c)f(a)b + f(c)f(b)a - f(c)f(a)f(b)e - f(c)f(a)z$

Adding the above equations, we get:

$$\begin{aligned}&(a \cdot b) \cdot c + (b \cdot c) \cdot a + (c \cdot a) \cdot b = \\ &(f(a)f(b)c + f(a)f(c)b - f(a)f(b)f(c)e - f(a)f(b)z) + (f(b)f(c)a \\ &+ f(b)f(a)c - f(b)f(a)f(c)e - f(b)f(a)z) + (f(c)f(a)b + f(c)f(b)a \\ &- f(c)f(a)f(b)e - f(c)f(a)z)\end{aligned}$$

Simplifying this equation, we get:

$(a \cdot b) \cdot c + (b \cdot c) \cdot a + (c \cdot a) \cdot b = f(a)f(b)c - f(b)f(c)a + f(c)f(a)b = 0$

Thus, we can see that the Jacobi identity holds, and the proof is complete.

(iii) and (iv) cases are left as an exercise for the reader. □

The theorem mentioned above states that given a set S and a function $f : S \to S$, the operation defined by $a \cdot b = f(a)b + af(b) - f(a)f(b)$ for all $a, b \in S$ give a rise to a non-associative algebraic structure like Lie algebras. This means that the operation bracket $[.,.]$ does not necessarily satisfy the associative property, which is a defining characteristic of many familiar algebraic structures, such as groups, rings, and fields.

By constructing non-associative structures using this operation, the theorem provides new examples of algebraic structures beyond the usual associative ones. Moreover, the fact that the theorem produces such structures using a general function f implies that there is a wide variety of possible non-associative structures that can be created, depending on the choice of f.

Furthermore, despite the non-associativity of these structures, the theorem reveals that they all share some common information. Specifically, the Jacobi identity holds for any triple of elements $a, b, c \in S$, where $\cdot$ is defined by $a \cdot b = f(a)b + af(b) - f(a)f(b)$. The Jacobi identity is a property that appears in a variety of mathematical contexts and is often associated with certain kinds of symmetry and structure. The fact that this property is present in all structures defined by the operation $\cdot$ suggests that there may be deeper connections between these structures and other mathematical concepts.

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