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A Logistic Black-Scholes Partial Differential Equation with Stochastic Volatility, Transaction Costs and Jumps

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Abstract

In this paper, we introduce a new differential form of an asset price. This form is proposed by considering various factors such as demand and supply on the asset, stochastic volatility, transaction costs and jumps. The new differential form extends an original logistic geometric Brownian motion by adding transaction cost and jump terms. Moreover, we find a solution for the asset price related to the proposed

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form. Furthermore, we derive Black-Scholes partial differential equations based on the proposed price process.

1 Introduction

An option is a financial contract in which investors can take benefits in different ways such as hedging, speculation and arbitrage. The benefits are taken by transferring risks form the option holder to the option seller. However, to do this agreement, the option holder has to pay a premium instead of taking risks. Therefore, the fair premium of an option has gained interest by many researchers such as Atzberger [1], Belze et al. [2] and Ghanadian et al. [3]. The well-known model for pricing the option is the Black-Scholes model which was introduced by Black and Scholes [4] and Merton [5]. This model is based on a partial differential equation of two independent variables: underlying asset price and time. However, a few assumptions are required in the model such as European-style options, no transaction costs, constant volatility and illiquid market. Also, it is focused when the underlying asset price follows a geometric Brownian motion. Over a few decades, many researchers have concentrated on developing the model in various directions; for example, Amster et al. [6] relaxed the assumption of no transaction costs and then used the hedging technique to find a new option pricing model. In another direction, it is believed that positive information may affect demand and supply of an asset. This leads to an overbought causing a rapid increase in the asset price. With this effect, Onyango [7] introduced excess demand functions, which can be used to compute an equilibrium price, and also introduced a new differential form of asset price related to the equilibrium price. This asset price process is called a logistic geometric Brownian motion. Under his differential form, Onyango assumed various assumptions such as no transaction costs, constant volatility and no dividend. After that, some researchers have tried to develop the original logistic geometric Brownian motion by reducing several assumptions and provided a differential equation model associated to their proposed process. In 2018, Nyakinda [8] derived a differential equation model associated to the price process given in [7] by considering transaction costs. In the same year, Nyakinda [9] replaced the assumption of constant volatility in the original logistic motion by a stochastic volatility satisfying a geometric Brownian motion.

For an illiquid market, the investor's trading affects the asset price due to a low trading volume. The impact of this trading is called a price impact which can be referred as the correlation between trading and the change in the price of the asset. In 2005, Liu and Yong [10] proposed a price impact function in an illiquid market and provided a differential form of risky asset with a price impact term. In 2013, El-Khatib and Hatemi-J [11] extended Liu and Yong model by adding a jump term due to changes in the asset price. In 2019, Mulambula et al. [12] proposed a logistic geometric Brownian motion with a jump and also provided a differential equation model under the given price process.

In this paper, we introduce a new logistic geometric Brownian motion with a transaction cost and a jump. We also provide the partial differential equations of the option price corresponding to the proposed process. The models are derived under the assumptions of non-constant volatility and stochastic volatility. In Section 2, we introduce the differential form of the new logistic geometric Brownian motion and investigate the price process. In Section 3, we formulate a jump diffusion model based on the proposed motion in case of non-constant and stochastic volatility.

2 Logistic Geometric Brownian Motion

In this section, we introduce a new differential form of the asset price following logistic geometric Brownian motion with stochastic volatility, transaction costs and jumps. A logistic geometric Brownian motion of asset price was first introduced by Onyango who gave the process as follows:

$$dS_t = \mu S_t(S^* - S_t)dt + \sigma S_t(S^* - S_t)dB_t, \tag{2.1}$$

where S_t is the stock price at time t, S^* is the equilibrium price, μ is the drift rate of asset, σ is the constant volatility and B_t is the standard Brownian motion. In 2018, Nyakinda [8] derived a logistic non-linear Black-Scholes-Merton partial differential equation when S_t satisfies the process in equation (2.1). He also considered a transaction cost in his derivation. Moreover, it is well-known that the volatility of an asset price is an important factor affecting the price. Therefore, some models focus on the volatility depending on several factors, such as stock price and time. Furthermore, the process has been developed under more general assumptions such as stochastic volatility and jump of the asset price. In 2018, Nyakinda [9] reduced the constant volatility assumption in the original logistic motion by assuming that S_t satisfies a logistic geometric Brownian motion being of the form:

$$dS_t = \mu S_t(S^* - S_t)dt + \sigma_t S_t(S^* - S_t)dB_1(t),$$

where σ_t is the stochastic volatility satisfying the following geometric Brownian motion:

$$d\sigma_t = \mu_\sigma \sigma_t dt + \nu_\sigma \sigma_t dB_2(t), \tag{2.2}$$

where μ_σ and ν_σ are the mean and variance of asset volatility, respectively, and $B_2(t)$ is the standard Brownian motion. In 2019, Mulambula et al. [12] proposed a logistic geometric Brownian motion with a jump term and also derived the non-linear differential equation. The stochastic differential form is given as:

$$dS_t = (\mu - \lambda k)S_t(S^* - S_t)dt + \sigma S_t(S^* - S_t)dB_t + S_t(S^* - S_t)(q_t - 1)dN_t, \tag{2.3}$$

where μ is the drift rate, λ is the rate at which the jumps happen, q_t is absolute price jump size, k is the average jump size measured as a proportional increase in the asset price, which is the average of $q_t - 1$, N_t is the Poisson process generating jumps. The parameter N_t is assumed to be λt. However, the volatility σ in this process is still assumed to be a constant.

In this work, we combine the ideas of [13], [11], [12] and [9] to construct a nonlinear Black-Scholes partial differential equation. This model is derived based on more general assumptions. The differential form of the risky asset S_t proposed is as follows:

$$\begin{aligned} dS_t = &(\mu - \lambda k)S_t(S^* - S_t)dt + \sigma_t S_t(S^* - S_t)dB_1(t) + S_t(S^* - S_t)(q_t - 1)dN_t \\ &+ \kappa(S_t, t)S_t(S^* - S_t)d\theta_t, \end{aligned}$$

where σ_t is the stochastic volatility defined in equation (2.2), $B_1(t)$ is standard Brownian motion, $\kappa(S_t, t)$ is the transaction cost and θ_t is the number of shares of the risky asset. We assume that the number of shares of the risky asset satisfies the following condition:

$$d\theta_t = \eta_t dt + \zeta_t\big(dB_1(t) + bdM_t\big),$$

where η_t and ζ_t are adapted processes, b is a real constant, $M_t = N_t - \lambda t$ is the compensated Poisson process associated to a Poisson process $(N_t)_{t\in[0,T]}$ with intensity λ. The equilibrium price S^* is assumed to be a positive constant. Therefore, we can write dS_t as

$$\begin{aligned} dS_t = &\Big[(\mu - \lambda k)S_t(S^* - S_t) + \kappa(S_t, t)S_t(S^* - S_t)\eta_t - \lambda\kappa(S_t, t)S_t(S^* - S_t)b\zeta_t\Big]dt \\ &+ \Big[\sigma_t S_t(S^* - S_t) + \kappa(S_t, t)S_t(S^* - S_t)\zeta_t\Big]dB_1(t) \\ &+ \Big[S_t(S^* - S_t)(q_t - 1) + \kappa(S_t, t)S_t(S^* - S_t)b\zeta_t\Big]dN_t \end{aligned}$$

which can also br rewritten as

$$dS_t = p(S_t,t)dt + u(S_t,t)dB_1(t) + v(S_t,t)dN_t, \tag{2.4}$$

where

$$\begin{aligned} p(S_t,t) &= S_t(S^* - S_t)\Big((\mu - \lambda k) + \kappa(S_t,t)\eta_t - \lambda\kappa(S_t,t)b\zeta_t\Big) \\ u(S_t,t) &= S_t(S^* - S_t)\Big(\sigma_t + \kappa(S_t,t)\zeta_t\Big) \\ v(S_t,t) &= S_t(S^* - S_t)\Big((q_t - 1) + \kappa(S_t,t)b\zeta_t\Big). \end{aligned}$$

Consequently, we can write dS_t in the above equation as

$$\frac{dS_t}{S_t(S^* - S_t)} = \frac{p(S_t,t)dt + u(S_t,t)dB_1(t) + v(S_t,t)dN_t}{S_t(S^* - S_t)}. \tag{2.5}$$

Starting from the above equation, we can investigate the price process by using partial fraction technique and Itô lemma. The result is provided in the following theorem:

Theorem 2.1. *Suppose that S_t satisfies the differential form in equation* (2.4) *and is positive for all $t \geq 0$. Then*

$$S_t = \begin{cases} \dfrac{S^* S_0}{S_0 - \big|S_0 - S^*\big| \exp(C(t))}, & S_t > S^* \\ \dfrac{S^* S_0}{S_0 + \big|S_0 - S^*\big| \exp(C(t))}, & S_t < S^*, \end{cases}$$

where

$$\begin{aligned} C(t) = \frac{1}{2}\int_0^t &\Big[\frac{1}{S_s^2} - \frac{1}{(S_s - S^*)^2}\Big] u^2(S_s,s)ds \\ &+ \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \int_0^t \Big[\frac{1}{S_s^n} - \frac{1}{(S_s - S^*)^n}\Big] v^n(S_s,s)dN_s \\ &+ S^* \int_0^t \Big[\frac{p(S_s,s)ds + u(S_s,s)dB_1(s) + v(S_s,s)dN_s}{S_s(S^* - S_s)}\Big]. \end{aligned}$$

Proof. Equation (2.5) can be simplified to;

$$\frac{1}{S^*}\left[\frac{1}{S_t - S^*} - \frac{1}{S_t}\right] dS_t = \frac{p(S_t,t)dt + u(S_t,t)dB_1(t) + v(S_t,t)dN_t}{S_t(S^* - S_t)}. \tag{2.6}$$

By applying the multidimensional Itô lemma, we have

$$d\ln(S_t) = \frac{1}{S_t}dS_t - \frac{1}{2S_t^2}(dS_t)^2 + \frac{1}{3S_t^3}(dS_t)^3 - \frac{1}{4S_t^4}(dS_t)^4 - ...$$

which implies that

$$\begin{aligned}\frac{1}{S_t}dS_t &= d\ln(S_t) + \frac{1}{2S_t^2}(dS_t)^2 - \frac{1}{3S_t^3}(dS_t)^3 + \frac{1}{4S_t^4}(dS_t)^4 - ...\\ &= d\ln(S_t) + \frac{1}{2S_t^2}\Big(u^2(S_t,t)dt + v^2(S_t,t)dN_t\Big) - \frac{1}{3S_t^3}v^3(S_t,t)dN_t\\ &\quad + \frac{1}{4S_t^4}v^4(S_t,t)dN_t + ... \end{aligned} \tag{2.7}$$

Similarly, applying the multidimensional Itô lemma, we obtain

$$\begin{aligned}\frac{1}{S_t - S^*}dS_t &= d\ln(S_t - S^*) + \frac{1}{2(S_t - S^*)^2}\Big(u^2(S_t,t)dt + v^2(S_t,t)dN_t\Big)\\ &\quad - \frac{1}{3(S_t - S^*)^3}v^3(S_t,t)dN_t + \frac{1}{4(S_t - S^*)^4}v^4(S_t,t)dN_t - ... \end{aligned} \tag{2.8}$$

From equations (2.6), (2.7) and (2.8), we obtain

$$\begin{aligned} d\ln(S_t - S^*) - d\ln(S_t) &= \frac{1}{2}\Big[\frac{1}{S_t^2} - \frac{1}{(S_t - S^*)^2}\Big]u^2(S_t,t)dt\\ &\quad + \sum_{n=2}^{\infty}\frac{(-1)^n}{n}\Big[\frac{1}{S_t^n} - \frac{1}{(S_t - S^*)^n}\Big]v^n(S_t,t)dN_t\\ &\quad + S^*\left[\frac{p(S_t,t)dt + u(S_t,t)dB_1(t) + v(S_t,t)dN_t}{S_t(S^* - S_t)}\right]. \end{aligned}$$

Integrating this equation over the interval $[0,t]$, we get

$$\ln\left|\frac{S_t - S^*}{S_t}\right| = \ln\left|\frac{S_0 - S^*}{S_0}\right| + C(t), \tag{2.9}$$

where

$$\begin{aligned} C(t) &= \frac{1}{2}\int_0^t \Big[\frac{1}{S_s^2} - \frac{1}{(S_s - S^*)^2}\Big]u^2(S_s,s)ds\\ &\quad + \sum_{n=2}^{\infty}\frac{(-1)^n}{n}\int_0^t \Big[\frac{1}{S_s^n} - \frac{1}{(S_s - S^*)^n}\Big]v^n(S_s,s)dN_s\\ &\quad + S^*\int_0^t \Big[\frac{p(S_s,s)ds + u(S_s,s)dB_1(s) + v(S_s,s)dN_s}{S_s(S^* - S_s)}\Big]. \end{aligned}$$

By deriving the above equation for the cases $S_t > S^*$ and $S_t < S^*$, the proof of the theorem is complete. □

Corollary 2.2. *Suppose that S_t satisfies the differential form in equation* (2.3) *and is positive for all $t \geq 0$. Then*

$$S_t = \begin{cases} \dfrac{S^* S_0}{S_0 - \left|S_0 - S^*\right| \exp(C(t))}, & S_t > S^* \\ \dfrac{S^* S_0}{S_0 + \left|S_0 - S^*\right| \exp(C(t))}, & S_t < S^*, \end{cases}$$

where

$$C(t) = \frac{1}{2}\int_0^t \sigma^2 S^*(S^* - 2S_s)ds + \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \int_0^t \Big((S^* - S_s)^n - S_s^n\Big)(q_s - 1)^n dN_s$$
$$+ S^* \int_0^t \Big[(\mu - \lambda k)ds + \sigma dB_s + (q_s - 1)dN_s\Big].$$

Proof. The proof follows immediately by replacing $\kappa(S_s, s) = 0$ in the above theorem. □

Remark 2.3. *From equation* (2.9), *note that if $S_0 > S^*$ and $S_t > S^*$ for $t > 0$, then the equation can be rewritten as*

$$\ln\left(\frac{S_t}{S_0}\right) = \ln\left(\frac{S_t - S^*}{S_0 - S^*}\right) - C(t).$$

From the above equation, the expected log-return equals the expected log-return adjusted by S^* minus $E(C(t))$. For example, if S_t satisfies equation (2.1), then

$$C(t) = \mu t + \sigma B(t) + \frac{\sigma^2}{2}\int_0^t (S^* - 2S_s)ds.$$

Therefore,

$$\mathbb{E}(R_t) = \mathbb{E}(R_t^*) - \mu t - \frac{\sigma^2 S^* t}{2} - \sigma^2 \int_0^t \mathbb{E}(S_s)ds,$$

where R_t and R_t^* are the log-return and the log-return adjusted by S^*, respectively.

3 Derivation of Logistic Black-Scholes Partial Differential Equation

In this section, we derive generalized versions of logistic Black-Scholes partial differential equation in two cases. With the assumption of non-constant volatility, the equation is derived based on hedging technique in section 3.1. For the assumption of stochastic volatility, it is investigated via the arbitrage approach in section 3.2. In the above two cases, transaction cost and price jump are also considered.

3.1 Logistic Black-Scholes Equation with Non-constant Volatility

In this section, the Logistic Black-Scholes equation is derived when the asset price S_t follows the proposed process in equation (2.4) and its volatility is not assumed to be a constant. This model is derived based on a hedging approach consisting of several steps. To investigate the equation, we first need an Itô formula of a function of an asset price associated to equation (2.4). The statement and proof are given as follows:

Proposition 3.1. *Let $f(S_t,t)$ be at least a twice differentiable function where S_t satisfies equation (2.4). If the asset price jumps from S_t to q_tS_t in the small time interval dt, then the differential of $f(S_t,t)$ is given by*

$$df(S_t,t) = \frac{\partial f(S_t,t)}{\partial t}dt + p(S_t,t)\frac{\partial f(S_t,t)}{\partial S_t}dt + u(S_t,t)\frac{\partial f(S_t,t)}{\partial S_t}dB_1(t) + \frac{1}{2}u(S_t,t)^2\frac{\partial^2 f(S_t,t)}{\partial S_t^2}dt + \Big[f(q_tS_t,t) - f(S_t,t)\Big]dN_t.$$

Proof. Let $f(S_t,t)$ be at least a twice differentiable function. By Taylor's expansion and the properties that $(dt)^2 = 0$ and $dS_tdt = 0$, we have

$$df(S_t,t) = \frac{\partial f(S_t,t)}{\partial t}dt + \frac{\partial f(S_t,t)}{\partial S_t}dS_t + \frac{1}{2}\frac{\partial^2 f(S_t,t)}{\partial S_t^2}(dS_t)^2.$$

Substituting equation (2.4) into the above equation, we obtain

$$\begin{aligned} df(S_t,t) &= \frac{\partial f(S_t,t)}{\partial t}dt + p(S_t,t)\frac{\partial f(S_t,t)}{\partial S_t}dt \\ &+ u(S_t,t)\frac{\partial f(S_t,t)}{\partial S_t}dB_1(t) + \frac{1}{2}u(S_t,t)^2\frac{\partial^2 f(S_t,t)}{\partial S_t^2}dt \\ &+ \left[v(S_t,t)\frac{\partial f(S_t,t)}{\partial S_t} + \frac{1}{2}v(S_t,t)^2\frac{\partial^2 f(S_t,t)}{\partial S_t^2}\right]dN_t. \end{aligned} \tag{3.10}$$

The last term in equation (3.10) is considered as the difference in the option value when a jump occurs. By the assumption, the last term can be written as $\Big[f(q_tS_t,t) - f(S_t,t)\Big]dN_t$; that is,

$$\begin{aligned} df(S_t,t) = &\frac{\partial f(S_t,t)}{\partial t}dt + p(S_t,t)\frac{\partial f(S_t,t)}{\partial S_t}dt + u(S_t,t)\frac{\partial f(S_t,t)}{\partial S_t}dB_1(t) \\ &+ \frac{1}{2}u(S_t,t)^2\frac{\partial^2 f(S_t,t)}{\partial S_t^2}dt + \Big[f(q_tS_t,t) - f(S_t,t)\Big]dN_t. \end{aligned}$$

The proof is now complete. □

Now, we are ready to construct the option pricing model using a hedging technique. The steps of this approach consist of the following. First, we set up a self-financing portfolio Π with a long position in the option $f(S_t,t)$ and short position in some quantity Δ of the underlying asset S_t; that is,

$$\Pi = f(S_t,t) - \Delta S_t.$$

Also, by the idea of hedging, this portfolio is supposed to be risk-free.
As time changes from t to $t+dt$, the self-financing portfolio means that the change in the value of the portfolio is due only to changes in the value of the assets. This assumption implies that

$$d\Pi = df(S_t,t) - \Delta dS_t.$$

Next, by the risk-free property of the portfolio, we choose Δ in the equation so that the term relating to randomness is zero.
Finally, the portfolio must earn the risk-free interest rate due to the risk-free property of the portfolio. Thus

$$d\Pi = r\Pi dt,$$

where r is the risk-free interest rate. The model is provided in the following Theorem:

Theorem 3.2. *Let $f(S_t,t)$ be an option price at time t. Suppose that S_t satisfies equation (2.4). Then the partial differential equation for the European call option price is as follows:*

$$\frac{\partial f(S_t,t)}{\partial t} - S_t\Big[(S^*-S_t)\Big(\lambda[q_t-1]+\kappa(S_t,t)b\zeta_t\lambda\Big)-r\Big]\frac{\partial f(S_t,t)}{\partial S_t} + \frac{1}{2}\Big[\sigma_t S_t(S^*-S_t)+\kappa(S_t,t)S_t(S^*-S_t)\zeta_t\Big]^2\frac{\partial^2 f(S_t,t)}{\partial S_t^2} - rf(S_t,t) + \lambda\mathbb{E}\Big[f(q_tS_t,t)-f(S_t,t)\Big]=0.$$

Proof. First, we construct a self-financing portfolio Π. Consider the situation that we buy one option at $f(S_t,t)$ and hedge with Δ shares of S_t, the portfolio Π will be set as

$$\Pi = f(S_t,t) - \Delta S_t, \tag{3.11}$$

where S_t is the underlying asset price, Δ is number of shares of S_t and $f(S_t,t)$ is the option price. By assuming self-financing portfolio, the change in the portfolio value over dt is

$$d\Pi = df(S_t,t) - \Delta dS_t. \tag{3.12}$$

Substituting (2.4) and Proposition (3.1) into (3.12), we get

$$\begin{aligned} d\Pi &= df(S_t,t) - \Delta dS_t \\ &= \left(\frac{\partial f(S_t,t)}{\partial S_t}-\Delta\right)\Big[p(S_t,t)dt+u(S_t,t)dB_1(t)\Big] + \frac{\partial f(S_t,t)}{\partial t}dt \\ &\quad + \frac{1}{2}u(S_t,t)^2\frac{\partial^2 f(S_t,t)}{\partial S_t^2}dt + \Big[f(q_tS_t,t)-f(S_t,t)\Big]dN_t - \Delta v(S_t,t)dN_t. \end{aligned} \tag{3.13}$$

Next, we consider the risk-free property of this portfolio. Thus the term involving the Brownian motion $dB_1(t)$ is zero. This implies that

$$\Delta = \frac{\partial f(S_t,t)}{\partial S_t}. \tag{3.14}$$

Substituting (3.14) into (3.13), we have

$$\begin{aligned} d\Pi &= \frac{\partial f(S_t,t)}{\partial t}dt + \frac{1}{2}u(S_t,t)^2\frac{\partial^2 f(S_t,t)}{\partial S_t^2}dt \\ &\quad + \Big[f(q_tS_t,t)-f(S_t,t)\Big]dN_t - \frac{\partial f(S_t,t)}{\partial S_t}v(S_t,t)dN_t. \end{aligned} \tag{3.15}$$

Finally, by the risk-free condition, the portfolio earns the risk-free rate. We have the equation $d\Pi = r\Pi dt$. By (3.11), (3.15) and (3.14), the equation becomes

$$\frac{\partial f(S_t,t)}{\partial t}dt + \frac{1}{2}u(S_t,t)^2\frac{\partial^2 f(S_t,t)}{\partial S_t^2}dt + \Big[f(q_tS_t,t) - f(S_t,t)\Big]dN_t$$
$$-\frac{\partial f(S_t,t)}{\partial S_t}v(S_t,t)dN_t = r\Big(f(S_t,t) - \frac{\partial f(S_t,t)}{\partial S_t}S_t\Big)dt.$$

Taking the expectation to the above equation, we get

$$\frac{\partial f(S_t,t)}{\partial t} - S_t\Big[(S^* - S_t)\Big(\lambda[q_t - 1] + \kappa(S_t,t)b\zeta_t\lambda\Big) - r\Big]\frac{\partial f(S_t,t)}{\partial S_t}$$
$$+\frac{1}{2}u(S_t,t)^2\frac{\partial^2 f(S_t,t)}{\partial S_t^2} - rf(S_t,t) + \lambda\mathbb{E}\Big[f(q_tS_t,t) - f(S_t,t)\Big] = 0. \quad (3.16)$$

The proof is now complete. □

From equation (3.16), note that if there is no transaction cost term, then equation (3.16) reduces to equation (34) given in [12]. In the next section, we extend the Black-Scholes model by considering a random part of the asset volatility. The stochastic volatility is assumed to satisfy a geometric Brownian motion given in (2.2).

3.2 Derivation of logistic Black-Scholes partial differential equation with stochastic volatility, transaction costs and jumps

In this section, we derive the stochastic differential equation of the option price when the stochastic volatility is assumed to satisfy a geometric Brownian motion in equation (2.2). We derive this model based on an arbitrage technique consisting of several steps. First, we construct a self-financing portfolio V_t with the risk-free asset and the risky asset. By the idea of this method, we assume that, at each time t, the value of the option $f(S_t,\sigma_t,t)$ must be equal to the value of the self-financing portfolio V_t and so are their differential forms. Thus the differential form of portfolio V_t and option price $f(S_t,\sigma_t,t)$ are also investigated. In the last step, the coefficients in the differential forms are compared in both random and non-random parts.

Let T be a maturity date and let $(V_t)_{t\in[0,T]}$ be a wealth process of a self-financing portfolio, A_t is the risk-free asset, $(\psi_t)_{t\in T}$ and θ_t denote the number of units invested in the risk-free and risky assets, respectively. Then the value of the portfolio is given by

$$V_t = \psi_t A_t + \theta_t S_t.$$

Assume that the price of the risk-free asset is

$$dA_t = r(S_t, t)A_t dt, \quad t \in [0, T].$$

By the self-financing assumption, this implies that

$$dV_t = \psi_t dA_t + \theta_t dS_t.$$

By the above equations and (2.4), we get

$$\begin{aligned} dV_t &= \left(\frac{V_t - \theta_t S_t}{A_t}\right)\Big(r(S_t, t)A_t dt\Big) + \theta_t dS_t \\ &= \Big[r(S_t, t)V_t - r(S_t, t)\theta_t S_t + \theta_t p(S_t, t)\Big]dt \\ &\quad + \theta_t u(S_t, t)dB_1(t) + \theta_t v(S_t, t)dN_t. \end{aligned} \tag{3.17}$$

In order to prove the main theorem, we need a multivariate Itô lemma for jump-diffusion processes in the case that S_t satisfies equation (2.4). This lemma aims to find a differential form of option price $f(S_t, \sigma_t, t)$. We prove Proposition (3.3) by combining an idea in Itô lemma for jump-diffusion processes in [11] and multivariate Itô lemma in [14]. Our proposition is stated as follows:

Proposition 3.3. *Let $f(S_t, \sigma_t, t)$ be the price of a European option and continuously twice differentiable function where S_t and σ_t satisfy equations* (2.4) *and* (2.2)*, respectively. If the asset price jumps from S_t to $q_t S_t$ in the small time interval dt, then the differential of $f(S_t, \sigma_t, t)$ is given by*

$$\begin{aligned} df(S_t, \sigma_t, t) &= \frac{\partial f(S_t, \sigma_t, t)}{\partial S_t} k(S_t, \sigma_t, t) + \frac{\partial f(S_t, \sigma_t, t)}{\partial \sigma_t}\Big[\mu_\sigma \sigma_t dt + \nu_\sigma \sigma_t dB_2(t)\Big] \\ &\quad + \frac{\partial f(S_t, \sigma_t, t)}{\partial t} dt + \frac{\partial^2 f(S_t, \sigma_t, t)}{\partial S_t \partial \sigma_t} u(S_t, t)\nu_\sigma \sigma_t \rho dt \\ &\quad + \frac{1}{2}\frac{\partial^2 f(S_t, \sigma_t, t)}{\partial S_t^2} u(S_t, t)^2 dt + \frac{1}{2}\frac{\partial^2 f(S_t, \sigma_t, t)}{\partial \sigma_t^2} \nu_\sigma^2 \sigma_t^2 dt \\ &\quad + \Big[f(q_t S_t, \sigma_t, t) - f(S_t, \sigma_t, t)\Big] dN_t, \end{aligned} \tag{3.18}$$

where $k(S_t, \sigma_t, t) = dS_t - v(S_t, t)dN_t$.

Proof. By Taylor expansion, we have

$$\begin{aligned} df(S_t,\sigma_t,t) =& \frac{\partial f(S_t,\sigma_t,t)}{\partial S_t}dS_t + \frac{\partial f(S_t,\sigma_t,t)}{\partial \sigma_t}d\sigma_t + \frac{\partial f(S_t,\sigma_t,t)}{\partial t}dt \\ &+ \frac{1}{2}\frac{\partial^2 f(S_t,\sigma_t,t)}{\partial S_t^2}(dS_t)^2 + \frac{\partial^2 f(S_t,\sigma_t,t)}{\partial S_t \partial \sigma_t}dS_t d\sigma_t \\ &+ \frac{1}{2}\frac{\partial^2 f(S_t,\sigma_t,t)}{\partial \sigma_t^2}(d\sigma_t)^2. \end{aligned} \tag{3.19}$$

Next, we compute $(dS_t)^2, dS_t d\sigma_t$ and $(d\sigma_t)^2$. We apply Itô multiplication table for the Poisson process into $(dS_t)^2, dS_t d\sigma_t$ and $(d\sigma_t)^2$. We get

$$(dS_t)^2 = u(S_t,t)^2dt + v(S_t,t)^2dN_t,\ dS_t d\sigma_t = u(S_t,t)\nu_\sigma \sigma_t \rho dt,\ (d\sigma_t)^2 = \nu_\sigma^2 \sigma_t^2 dt.$$

By plugging equations (2.2), (2.4) and the above three equations into equation (3.19) and applying the assumption that the jump term dN_t can be written as $\big(f(q_t S_t,\sigma_t,t) - f(S_t,\sigma_t,t)\big)dN_t$ into such an equation, our proof is complete. □

The following theorem gives the logistic Black-Scholes partial differential equation for option pricing with stochastic volatility, transaction costs and jumps.

Theorem 3.4. *Let $f(S_t,\sigma_t,t)$ be a continuously twice differentiable function. Suppose that S_t and σ_t satisfy equations* (2.4) *and* (2.2), *respectively. Then the partial differential equation for the European call option price is as follows:*

$$\begin{aligned} r(S_t,t)V_t - r(S_t,t)\theta_t S_t + \theta_t p(S_t,t) =& \frac{\partial f(S_t,\sigma_t,t)}{\partial S_t}p(S_t,t) + \frac{\partial f(S_t,\sigma_t,t)}{\partial t} \\ &+ \frac{1}{2}\frac{\partial^2 f(S_t,\sigma_t,t)}{\partial S_t^2}u(S_t,t)^2 \end{aligned} \tag{3.20}$$

with the terminal condition $f(S_T,\sigma_T,T) = h(S_T)$.

Proof. By the arbitrage technique, at each time t, the value of the option $f(S_t,\sigma_t,t)$ is equal to the value of the portfolio V_t; that is, $f(S_t,\sigma_t,t) = V_t$ for all $t \in [0,T]$. This implies that $dV_t = df(S_t,\sigma_t,t,)$ for all $t \in (0,T)$. By comparing the random part $dB_2(t)$ in (3.17) and (3.18), we have $\dfrac{\partial f(S_t,\sigma_t,t)}{\partial \sigma_t} = 0$. With this fact, we consider the given equation by comparing the coefficient term of dt in (3.17) and (3.18). We get the partial differential equation for the European call option price:

$$r(S_t,t)V_t - r(S_t,t)\theta_t S_t + \theta_t p(S_t,t) = \frac{\partial f(S_t,\sigma_t,t)}{\partial S_t} p(S_t,t) + \frac{\partial f(S_t,\sigma_t,t)}{\partial t} + \frac{1}{2}\frac{\partial^2 f(S_t,\sigma_t,t)}{\partial S_t^2} u(S_t,t)^2$$

with the terminal condition $f(S_T,\sigma_T,T) = h(S_T)$. □

In replicating portfolio V_t, there is the number of unit invested in the risky asset θ_t which depends on t. Due to the arbitrage approach, we compare the coefficients for $dB_1(t)$ and dN_t in equations (3.17) and (3.18). For the coefficient of $dB_1(t)$, we get

$$\theta_t S_t(S^* - S_t)\Big[\sigma_t + \kappa(S_t,t)\zeta_t\Big] = \frac{\partial f(S_t,\sigma_t,t)}{\partial S_t} S_t(S^* - S_t)\Big[\sigma_t + \kappa(S_t,t)\zeta_t\Big].$$

For the coefficient of dN_t, we similarly have

$$\theta_t S_t(S^* - S_t)\Big[(q_t - 1) + \kappa(S_t,t)b\zeta_t\Big] = \Big[f(q_t S_t,\sigma_t,t) - f(S_t,\sigma_t,t)\Big].$$

By comparing the coefficients of $dB_1(t)$ and dN_t, we obtain the different values of the number of shares θ_t. Thus we can not find the number of shares θ_t that lead to the value $V_T = h(S_T) = f(S_T,\sigma_T,T)$. However, we can find the number of shares θ_t that minimizes the difference between $h(S_T)$ and V_T when the correlation between $dB_1(t)$ and $dB_2(t)$ is zero. The result is stated in the following proposition:

Proposition 3.5. *Suppose that the the correlation between $dB_1(t)$ and $dB_2(t)$ is zero. Then the number of shares θ_t that minimizes the variance is given by*

$$\theta_t = \frac{\frac{\partial f(S_t,\sigma_t,t)}{\partial S_t} u(S_t,t)^2 + \lambda\Big(f(q_t S_t,\sigma_t,t) - f(S_t,\sigma_t,t)\Big)v(S_t,t)}{u(S_t,t)^2 + \lambda v(S_t,t)^2}.$$

Proof. In this proposition, we find the number of shares θ_t that minimizes the distance between payoff of option and portfolio at maturity date T; that is, we need to find the θ_t that solves the following problem:

$$\min_{\theta_t} \mathbb{E}\Big[(h(S_T) - V_T)^2\Big].$$

By using equations (3.17), (3.18) with $\dfrac{\partial f(S_t,\sigma_t,t)}{\partial \sigma_t} = 0$, we have

$$V_T = V_0 + \int_0^T \Big[r(S_t,t)V_t - r(S_t,t)\theta_t S_t + \theta_t p(S_t,t)\Big]dt + \int_0^T \theta_t u(S_t,t)dB_1(t) + \int_0^T \theta_t v(S_t,t)dN_t \tag{3.21}$$

and

$$\begin{aligned} f(S_T,\sigma_T,T) = f(S_0,\sigma_0,0) &+ \int_0^T \frac{\partial f(S_t,\sigma_t,t)}{\partial S_t}p(S_t,t)dt \\ &+ \int_0^T \frac{\partial f(S_t,\sigma_t,t)}{\partial S_t}u(S_t,t)dB_1(t) + \int_0^T \frac{\partial f(S_t,\sigma_t,t)}{\partial t}dt \\ &+ \frac{1}{2}\int_0^T \frac{\partial^2 f(S_t,\sigma_t,t)}{\partial S_t^2}u(S_t,t)^2 dt \\ &+ \int_0^T \Big[f(q_t S_t,\sigma_t,t) - f(S_t,\sigma_t,t)\Big]dN_t. \end{aligned} \tag{3.22}$$

From the terminal condition $f(S_T,\sigma_T,T) = h(S_T)$, (3.21) and (3.22), we have

$$\begin{aligned} h(S_T) - V_T = &\int_0^T \Big(\frac{\partial f(S_t,\sigma_t,t)}{\partial S_t} - \theta_t\Big)u(S_t,t)dB_1(t) \\ &+ \int_0^T \Big[f(q_t S_t,\sigma_t,t) - f(S_t,\sigma_t,t) - \theta_t v(S_t,t)\Big]dN_t. \end{aligned}$$

In the above equation, the term related to dt vanishes due to the equation (3.20). By the assumption that the correlation between $dB_1(t)$ and $dB_2(t)$ is zero, $dB_1(t)$ and dN_t are independent, we have $\mathbb{E}\Big[h(S_T)\cdot V_T\Big]$=0. Therefore,

$$\begin{aligned} \mathbb{E}\Big[\big(h(S_T) - V_T\big)^2\Big] &= \mathbb{E}\Big[\Big(\int_0^T \big(\frac{\partial f(S_t,\sigma_t,t)}{\partial S_t} - \theta_t\big)u(S_t,t)dB_1(t)\Big)^2\Big] \\ &\quad + \mathbb{E}\Big[\Big(\int_0^T \big(f(q_t S_t,\sigma_t,t) - f(S_t,\sigma_t,t) - \theta_t v(S_t,t)\big)dN_t\Big)^2\Big] \\ &= \mathbb{E}\Big[\int_0^T \Big(\big(\frac{\partial f(S_t,\sigma_t,t)}{\partial S_t} - \theta_t\big)u(S_t,t)\Big)^2 dt\Big] \\ &\quad + \mathbb{E}\Big[\int_0^T \lambda\Big(f(q_t S_t,\sigma_t,t) - f(S_t,\sigma_t,t) - \theta_t v(S_t,t)\Big)^2 dt\Big]. \end{aligned}$$

We will find θ_t that minimizes the variance by using the derivative test. From Leibniz integral rule, we can find a critical point by considering an equation:

$$-2\Big[\frac{\partial f(S_t,\sigma_t,t)}{\partial S_t}u(S_t,t)^2+\lambda\big(f(q_tS_t,\sigma_t,t)-f(S_t,\sigma_t,t)\big)v(S_t,t)\Big] \\ +2\theta_t\Big[u(S_t,t)^2+\lambda v(S_t,t)^2\Big]=0$$

which implies that

$$\theta_t=\frac{\dfrac{\partial f(S_t,\sigma_t,t)}{\partial S_t}u(S_t,t)^2+\lambda\Big(f(q_tS_t,\sigma_t,t)-f(S_t,\sigma_t,t)\Big)v(S_t,t)}{u(S_t,t)^2+\lambda v(S_t,t)^2}.$$

Thus we have a critical point. For the second derivative test, we have

$$2u(S_t,t)^2+2\lambda v(S_t,t)^2>0.$$

Hence, we get θ_t as required. □

4 Conclusions

An extended version of the differential form of an asset price satisfying a logistic geometric Brownian motion was introduced by adding terms related to transaction costs and jumps. Moreover, a solution for the proposed process was also derived. Furthermore, two logistic Black-Scholes partial differential equations were constructed when the asset volatility was assumed to be non-constant and stochastic, respectively. Our results differed from previous studies in which the asset price was more general and realistic than the previous ones. As a result, this research contributes knowledge to researchers in the area of Financial Mathematics.

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