Co-Lindelöf Open Sets

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Abstract

In this paper, we define a new type of open sets called co-Lindelöf open set (coL-open). Moreover, we obtain several results about co-Lindelöf open sets and study the relation among open set, coc-open set and coL-open. Furthermore, we define separation axioms via coL-open and prove some related results.

Key words and phrases: co-Lindelöf open sets, coL-$T_i$- spaces $i = 0, 1, 2$, coL-$D_i$- spaces $i = 0, 1, 2$.

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1 Introduction

One of the interesting problems in topology is to define weaker and stronger forms of open sets. When topologists construct a new type of open sets, several definitions, theorems, examples and open problems appear. For example, Alghour and Samarah [3] defined a co-compact open set as follows:

Definition 1.1. [3] A subset $A$ of a topological space $X$ is called a co-compact open set (in short, coc-open) if for every $x \in A$, there exists an open set $U \subseteq X$ and a compact subset $K$ of $X$ such that $x \in U - K \subseteq A$. The complement of a coc-open subset is called coc-closed. The family of all coc-open subsets of a topological space $(X, \tau)$ will be denoted by $\tau^k$.

Theorem 1.2. [3] Let $(X, \tau)$ be a space. Then the collection $\mathcal{B}^k(\tau) = \{U - K : U \in \tau \text{ and } K \text{ is a compact subset of } X\}$ forms a base for $\tau^k$.

For more papers about coc-compact spaces, see [1, 2, 5]. For terms and notations not explained in this paper, the reader is referred to [4].

Definition 1.3. A subset $A$ of a topological space $(X, \tau)$ is called a co-Lindelöf open set (in short, coL-open) if for every $x \in A$, there exists an open set $U \subseteq X$ and a Lindelöf subset $L$ of $X$ such that $x \in U - L \subseteq A$. The complement of a coc-open subset is called coc-closed. The family of all coc-open subsets of a topological space $(X, \tau)$ will be denoted by $\tau^L$.

Lemma 1.4. Let $(X, \tau)$ be a topological space. Then the collection $\tau^L$ forms a topology on $X$.

Proof. Let $V_1, V_2 \in \tau^L$ and let $x \in V_1 \cap V_2$. So there are open subsets $W_1, W_2$ of $X$ and Lindelöf subsets $L_1, L_2$ of $X$ with $x \in W_1 - L_1 \subseteq V_1$ and $x \in W_2 - L_2 \subseteq V_2$. Consider the subsets $W = W_1 \cap W_2$ and $L = L_1 \cup L_2$. Then $W$ is an open subset of $X$ and $L$ is Lindelöf subset of $X$ with $x \in W - L \subseteq V_1 \cap V_2$. Let $\{V_\alpha | \alpha \in \Delta\}$ be a family of coL-open subsets of $X$. For $x \in \bigcup_{\alpha \in \Delta} V_\alpha$, there is $\alpha^* \in \Delta$ such that $x \in V_{\alpha^*}$, but $V_{\alpha^*} = U - L$, where $U$ is an open subset of $X$ and $L$ is Lindelöf subset of $X$. So $x \in U - L \subseteq \bigcup_{\alpha \in \Delta} V_\alpha$.

Clearly $\phi, X \in \tau^L$ and hence the collection $\tau^L$ forms a topology on $X$. \qed

Lemma 1.5. Let $(X, \tau)$ be a space. Then $\mathcal{B}^L(\tau) = \{U - L : U \in \tau, L \text{ is a Lindelöf subset of } X\}$ forms a base for $\tau^L$.

Theorem 1.6. For a topological space $(X, \tau)$, we have $\tau \subseteq \tau^k \subseteq \tau^L$. 

In general, the converse of the second inclusion of the last theorem is not true as can be seen from the following example:

**Example 1.7.** Let \( X = \mathbb{N} \) and \( \tau = \{ \emptyset, X \} \cup \{ U_n : n \in \mathbb{N} \} \), where \( U_n = \{ n, n+1, n+2, \cdots \} \). Then \( \tau^k = \tau \cup \{ X - K : K \text{ finite} \} \) and \( \tau^L = \tau_{\text{dis}} \) since for \( U_n \in \tau \) and \( U_{n+1} \) a Lindelöf subset of \( X \), we have \( \{ n \} = U_n - U_{n+1} \).

**Definition 1.8.** A topological space \( (X, \tau) \) is called an \( LC \) space if every Lindelöf subset is closed.

**Theorem 1.9.** For a topological space \( (X, \tau) \), the following are equivalent:

1. \( (X, \tau) \) is LC,
2. \( \tau = B^L(\tau) \),
3. \( \tau = \tau^k = \tau^L \).

**Proof.** (1) \( \rightarrow \) (2) For each \( U \in \tau \), \( U - \phi \in B^L(\tau) \), and for \( U - L \in B^L(\tau) \), where \( U \in \tau \) and \( L \) is Lindelöf and hence closed. As a result, \( U - L \in \tau \).

(2) \( \rightarrow \) (3) Since \( \tau = B^L(\tau) \), \( B^L(\tau) \) is a base for \( \tau^L \). Therefore, \( \tau^L \subseteq \tau \).

(3) \( \rightarrow \) (1) Let \( L \) be Lindelöf subset of \( X \). Then \( X - L \in \tau^L = \tau \) and so \( L \) is LC.

**Definition 1.10.** A topological space \( (X, \tau) \) is called a \( p \)-space if the countable union of open subsets of \( X \) is open.

**Corollary 1.11.** If \( (X, \tau) \) is a \( T_2 \) \( p \)-space, then \( \tau = \tau^k = \tau^L \).

**Theorem 1.12.** If \( (X, \tau) \) is a hereditarily Lindelöf space, then \( \tau^L = \tau_{\text{dis}} \).

**Proof.** For \( x \in X \), \( X - \{ x \} \) is Lindelöf. But \( \{ x \} = X - (X - \{ x \}) \). Therefore, the result follows.

**Corollary 1.13.** If \( (X, \tau) \) is a hereditarily compact space, then \( \tau^L = \tau_{\text{dis}} \).

### 2 \ coL-\( T \)-spaces and coL-\( D \)-spaces

**Definition 2.1.** A space \( (X, \tau) \) is called a co-Lindelöf-\( T_0 \) space (in short, coL-\( T_0 \)-space) if for all \( x \neq y \in X \), there exists a coL-open set \( U \) containing one point but not the other.
It is clear that every $T_0$-space is a coL-$T_0$-space. The indiscrete topology with two points is an example of a coL-$T_0$-space which is not a $T_0$-space.

**Lemma 2.2.** If $A$ is coL-closed subset of $X$, then $(\tau|_A)^L = \tau^L|_A$.

**Proof.** $(\subseteq)$ Clear.

$(\supseteq)$ Let $V \in \tau^L|_A$ with $y \in V$. Then there is a $\tau^L$ subset $W$ of $X$ with $V = W \cap A$. Similarly for $W$, there is an open subset $U$ and a Lindelöf subset $L$ with $y \in U - L \subseteq W$. So $y \in (U \cap A) - (L \cap A)$. This means that $U \cap A \in \tau|_A$ and $L \cap A$ is Lindelöf in $A$. Consequently, the result follows.

**Definition 2.3.** A subset $A$ of a topological space $(X, \tau)$ is called a coL-set if $A = U - V$, for some $U, V \in \tau^L$.

**Definition 2.4.** A space $(X, \tau)$ is called a co-Lindelöf-$D_0$-space (in short, coL-$D_0$-space) if, for all $x \neq y \in X$, there exists a coL-$D_0$-set $U$ containing one point but not the other.

**Theorem 2.5.** A coL-closed subset of a coL-$D_0$-space $(X, \tau)$ is a coL-$D_0$-space.

**Proof.** Let $A$ be a coL-closed subset of $X$ and let $x \neq y \in A$. So there exists a coL-$D_0$-set $D = U - V$ with $U, V \in \tau^k$ such that $x \in D$ and $y \notin D$. Now, $x \in D \cap A = (U - V) \cap A = (A \cap U) - (L \cap A)$ but $A \cap U$ and $A \cap V \in (\tau|_A)^L = \tau^L|_A$. The result now follows.

**Theorem 2.6.** A space $(X, \tau)$ is a coL-$T_0$-space if and only if it is a coL-$D_0$-space.

**Proof.** $(\Rightarrow)$ This is clear since every proper coL-open subset of $X$ is a coL-$D$-set.

$(\Leftarrow)$ Let $x \neq y \in X$. Then there exists a coL-$D_0$-set $U$ containing $x$ with $U = U_1 - U_2$, where $U_1, U_2 \in \tau^L$; i.e., $x \in U_1$ and $x \notin U_2$. For $y$, we have the following cases:

(1) If $y \notin U_1$, then we are done.

(2) If $y \in U_1$ and $y \notin U_2$, then $U_2$ contains $y$ but not $x$.

**Definition 2.7.** Let $(X, \tau)$ be a space and $A \subseteq X$. The coL-closure of $A$ in $X$, denoted by $\overline{A}^{coL} = \cap\{B : B \text{ is a coL-closed in } X, A \subseteq B\}$.

**Theorem 2.8.** A space $(X, \tau)$ is a coL-$T_0$-space if and only if for all $x \neq y \in X$, we have $\overline{\{x\}}^{coL} \neq \overline{\{y\}}^{coL}$.
Proof. \((\Rightarrow)\) Let \(x \neq y \in X\). Then there exists a coL- open set \(U\) containing one point but not the other; say, \(x \in U\) and \(y \notin U\). Then \(X - U\) is a coL-closed set containing \(y\) and \(\{y\}^{coL} \subseteq X - U\). So \(x \notin \{y\}^{coL}\). Hence \(\{x\}^{coL} \neq \{y\}^{coL}\).

\((\Leftarrow)\) Let \(x \neq y \in X\). Clearly, \(X - \{y\}^{coL}\) is a coc-open set containing \(x\) but not \(y\). Hence \(X\) is coL-\(T_0\)-space.

\(\Box\)

**Definition 2.9.** A space \((X, \tau)\) is called a co-Lindelöf-\(T_1\)-space (in short, coL-\(T_1\)-space) if for all \(x \neq y \in X\), there exist coL-open sets \(U_x, V_y\) with \(\{U_x, V_y\} \cap \tau \neq \phi\) such that \(x \in U_x\), \(y \in V_y\) and \(y \notin U_x\), \(x \notin V_y\).

**Definition 2.10.** A space \((X, \tau)\) is called a co-Lindelöf-\(T_2\)-space (in short, coL-\(T_2\)-space) if for all \(x \neq y \in X\), there exist coL-open sets \(U_x, V_y\) with \(\{U_x, V_y\} \cap \tau \neq \phi\) such that \(x \in U_x\), \(y \in V_y\) and \(U_x \cap V_y = \phi\).

It is clear that if \((X, \tau)\) is a coL-\(T_1\)-space, then \((X, \tau^L)\) is \(T_1\)-space. In addition, every \(T_1\)-space is a coL-\(T_1\)-space. The converse need not be true as the following example shows:

**Example 2.11.** Let \(X = \mathbb{N}\) and \(\tau = \{\phi, X\} \cup \{U_n : n \in \mathbb{N}\}\), where \(U_n = \{n, n+1, n+2, \cdots\}\). Clearly, \((X, \tau)\) is not a \(T_1\)-space since for \(1, 2 \in X\) each open set containing 1 must contains 2. But \((X, \tau)\) is a coL-\(T_1\)-space and a coL-\(T_2\)-space since \(\tau^L = \tau\).

**Theorem 2.12.** A space \((X, \tau)\) is a coL-\(T_1\)-space if and only if every singleton is coL-closed.

**Definition 2.13.** A space \((X, \tau)\) is called co-Lindelöf-\(D_1\)-space (in short, coL-\(D_1\)-space) if for all \(x \neq y \in X\), there exist coL-\(D\)-sets \(U_x, V_y\) such that \(x \in U_x\), \(y \in V_y\) and \(y \notin U_x\), \(x \notin V_y\).

**Theorem 2.14.** A coL-closed subspace of a coL-\(D_1\)-space \((X, \tau)\) is coL-\(D_1\)-space.

**Definition 2.15.** A space \((X, \tau)\) is called co-Lindelöf-\(D_2\)-space (in short, coL-\(D_2\)-space) if for all \(x \neq y \in X\), there exist disjoint coL-\(D\)-sets \(U_x, V_y\) such that \(x \in U_x\), \(y \in V_y\) and \(y \notin U_x\), \(x \notin V_y\).

**Theorem 2.16.** Let \((X, \tau)\) be a topological space. Then

1. If \((X, \tau)\) is a coL-\(T_i\)-space, then \((X, \tau)\) is a coL-\(T_{i-1}\)-space for \(i = 1, 2\).

2. If \((X, \tau)\) is a coL-\(T_i\)-space, then \((X, \tau)\) is a coL-\(D_i\)-space for \(i = 1, 2\).
3. If \((X, \tau)\) is a coL-D\(_i\)-space, then \((X, \tau)\) is a coL-D\(_{i-1}\)-space for \(i = 1, 2\).

4. If \((X, \tau)\) is a coL-D\(_1\)-space, then \((X, \tau)\) is a coL-T\(_0\)-space.

5. \((X, \tau)\) is a coL-D\(_1\)-space if and only if \((X, \tau)\) is a coL-D\(_2\)-space.

Proof. We will prove 5.

\((\Rightarrow)\) Obvious.

\((\Leftarrow)\) For \(x \neq y \in X\), there exist coL-D-sets \(U_1, U_2\) with \(x \in U_1, y \notin U_1\) and \(y \in U_2, x \notin U_1\). Assume \(U_1 = V_1 - W_1, U_2 = V_2 - W_2\), where \(V_1, W_1, V_2, W_2 \in \tau^L\). Then, for \(x \notin U_2\), we have the following cases:

1. \(x \notin V_2\),
2. \(x \in V_2\) and \(x \in W_2\).

For (1), if \(x \notin V_2\), we have:

(i) \(y \notin V_1, x \in V_1 - W_1\). Then \(x \in V_1 - (V_2 \cup W_1)\) and \(y \in V_2 - W_2\), so \(y \in V_2 - (V_1 \cup W_2)\) and \((V_1 - (V_2 \cup W_1)) \cap (V_2 - (V_1 \cup W_2)) = \phi\).

(ii) \(y \in V_1\) and \(y \in W_1\), we have \(x \in U_1 - U_2, y \in U_2\) and \((U_1 - U_2) \cap U_2 = \phi\).

For (2) If \(y \in U_2 = V_2 - W_2\), \(x \in W_2\) and \((V_2 - W_2) \cap W_2 = \phi\), from (1) and (2) \((X, \tau)\) is a coL-D\(_2\)-space.

Theorem 2.17. A space \((X, \tau)\) is a coL-D\(_1\)-space if and only if \((X, \tau)\) is a coL-T\(_0\)-space and \(\text{int}_{coL}(A_x) \neq X\) for all \(x \in A_x \subseteq X\).

Proof. \((\Rightarrow)\) For \(x \in X\), there exists a coL-D-set \(O_x = U - V\) with \(U, V \in \tau^L\) and \(x \in O_x\). But \(U \neq X\). So \(\text{int}_{coL}(U) \neq X\). Hence the result follows.

\((\Leftarrow)\) For \(x \neq y \in X\), without loss of generality, there exists a coL-open set \(U\) containing \(x\) but not \(y\) and there exists coc-open set \(V\) containing \(y\) and \(\text{int}_{coL}(V) \neq X\). Hence \(y \in V - U\). Therefore, \((X, \tau)\) is a coLD\(_1\)-space.

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