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Co-Lindelöf Open Sets

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Abstract

In this paper, we define a new type of open sets called co-Lindelöf open set (coL-open). Moreover, we obtain several results about co-Lindelöf open sets and study the relation among open set, coc-open set and coL-open. Furthermore, we define separation axioms via coL-open and prove some related results.

Key words and phrases: co-Lindelöf open sets, coL-T_i- spaces $i = 0, 1, 2$, coL-D_i- spaces $i = 0, 1, 2$.
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1 Introduction

One of the interesting problems in topology is to define weaker and stronger forms of open sets. When topologists construct a new type of open sets, several definitions, theorems, examples and open problems appear. For example, Al ghour and Samarah [3] defined a co-compact open set as follows:

Definition 1.1. [3] *A subset A of a topological space X is called a co-compact open set (in short, coc-open) if for every $x \in A$, there exists an open set $U \subseteq X$ and a compact subset K of X such that $x \in U - K \subseteq A$. The complement of a coc-open subset is called coc-closed. The family of all coc-open subsets of a topological space (X, τ) will be denoted by τ^k.*

Theorem 1.2. [3] *Let (X, τ) be a space. Then the collection $\mathcal{B}^k(\tau) = \{U - K : U \in \tau$ and $K$is a compact subset of $X\}$ forms a base for τ^k.*

For more papers about coc-compact spaces, see [1, 2, 5]. For terms and notations not explained in this paper, the reader is referred to [4].

Definition 1.3. *A subset A of a topological space (X, τ) is called a co-Lindelöf open set (in short, coL-open) if for every $x \in A$, there exists an open set $U \subseteq X$ and a Lindelöf subset L of X such that $x \in U - L \subseteq A$. The complement of a coc-open subset is called coc-closed. The family of all coc-open subsets of a topological space (X, τ) will be denoted by τ^L.*

Lemma 1.4. *Let (X, τ) be a topological space. Then the collection τ^L forms a topology on X.*

Proof. Let $V_1, V_2 \in \tau^L$ and let $x \in V_1 \cap V_2$. So there are open subsets W_1, W_2 of X and Lindelöf subsets L_1, L_2 of X with $x \in W_1 - L_1 \subseteq V_1$ and $x \in W_2 - L_2 \subseteq V_2$. Consider the subsets $W = W_1 \cap W_2$ and $L = L_1 \cup L_2$. Then W is open subset of X and L is Lindelöf subset of X with $x \in W - L \subseteq V_1 \cap V_2$. Let $\{V_\alpha | \alpha \in \Delta\}$ be a family of coL-open subsets of X. For $x \in \bigcup_{\alpha \in \Delta} V_\alpha$, there is $\alpha^* \in \Delta$ such that $x \in V_{\alpha^*}$ but $V_{\alpha^*} = U - L$, where U is an open subset of X and L is Lindelöf subset of X. So $x \in U - L \subseteq \bigcup_{\alpha \in \Delta} V_\alpha$.

Clearly $\phi, X \in \tau^L$ and hence the collection τ^L forms a topology on X. □

Lemma 1.5. *Let (X, τ) be a space. Then*
$\mathcal{B}^L(\tau) = \{U - L : U \in \tau, L$ is a Lindelöf subset of $X\}$ forms a base for τ^L.

Theorem 1.6. *For a topological space (X, τ), we have $\tau \underset{\neq}{\subsetneq} \tau^k \underset{\neq}{\subsetneq} \tau^L$.*

In general, the converse of the second inclusion of the last theorem is not true as can be seen from the following example:

Example 1.7. *Let $X = \mathbb{N}$ and $\tau = \{\phi, X\} \cup \{U_n : n \in \mathbb{N}\}$, where $U_n = \{n, n+1, n+2, \cdots\}$. Then $\tau^k = \tau \cup \{X - K : K \quad \text{finite}\}$ and $\tau^L = \tau_{dis}$ since for $U_n \in \tau$ and U_{n+1} a Lindelöf subset of X, we have $\{n\} = U_n - U_{n+1}$.*

Definition 1.8. *A topological space (X, τ) is called an LC space if every Lindelöf subset is closed.*

Theorem 1.9. *For a topological space (X, τ), the following are equivalent:*

1. *(X, τ) is LC,*
2. *$\tau = \mathcal{B}^L(\tau)$,*
3. *$\tau = \tau^k = \tau^L$.*

Proof. (1) → (2) For each $U \in \tau$, $U - \phi \in \mathcal{B}^L(\tau)$, and for $U - L \in \mathcal{B}^L(\tau)$, where $U \in \tau$ and L is Lindelöf and hence closed. As a result, $U - L \in \tau$.
(2) → (3) Since $\tau = \mathcal{B}^L(\tau)$, $\mathcal{B}^L(\tau)$ is a base for τ^L. Therefore, $\tau^L \subseteq \tau$.
(3) → (1) Let L be Lindelöf subset of X. Then $X - L \in \tau^L = \tau$ and so L is LC. □

Definition 1.10. *A topological space (X, τ) is called a p-space if the countable union of open subsets of X is open.*

Corollary 1.11. *If (X, τ) is a T_2 p-space, then $\tau = \tau^k = \tau^L$.*

Theorem 1.12. *If (X, τ) is a hereditarily Lindelöf space, then $\tau^L = \tau_{dis}$.*

Proof. For $x \in X$, $X - \{x\}$ is Lindelöf. But $\{x\} = X - (X - \{x\})$. Therefore, the result follows.

□

Corollary 1.13. *If (X, τ) is a hereditarily compact space, then $\tau^L = \tau_{dis}$.*

2 coL-T-spaces and coL-D-spaces

Definition 2.1. *A space (X, τ) is called a co-Lindelöf-T_0- space (in short, coL-T_0-space) if for all $x \neq y \in X$, there exists a coL-open set U containing one point but not the other.*

It is clear that every T_0-space is a coL-T_0-space. The indiscrete topology with two points is an example of a coL-T_0-space which is not a T_0-space.

Lemma 2.2. *If A is coL-closed subset of X, then $(\tau\mid_A)^L = \tau^L\mid_A$.*

Proof. ($\subseteq$) Clear.
($\supseteq$) Let $V \in \tau^L\mid_A$ with $y \in V$. Then there is a τ^L subset W of X with $V = W \cap A$. Similarly for W, there is an open subset U and a Lindelöf subset L with $y \in U - L \subseteq W$. So $y \in (U \cap A) - (L \cap A)$. This means that $U \cap A \in \tau\mid_A$ and $L \cap A$ is Lindelöf in A. Consequently, the result follow. □

Definition 2.3. *A subset A of a topological space (X, τ) is called a coL-D-set if $A = U - V$, for some $U, V \in \tau^L$.*

Definition 2.4. *A space (X, τ) is called a co-Lindelöf-D_0- space (in short, coL-D_0-space) if, for all $x \neq y \in X$, there exists a coL-D-set U containing one point but not the other.*

Theorem 2.5. *A coL-closed subset of a coL-D_0-space (X, τ) is a coL-D_0-space.*

Proof. Let A be a coL-closed subset of X and let $x \neq y \in A$. So there exists a coL-D-set $D = U - V$ with $U, V \in \tau^k$ such that $x \in D$ and $y \notin D$. Now, $x \in D \cap A = (U - V) \cap A = (A \cap U) - (A \cap V)$ but $A \cap U$ and $A \cap V \in (\tau\mid_A)^L = \tau^L\mid_A$. The result now follows. □

Theorem 2.6. *A space (X, τ) is a coL-T_0-space if and only if it is a coL-D_0-space.*

Proof. ($\Rightarrow$) This is clear since every proper coL-open subset of X is a coL-D-set.
($\Leftarrow$) Let $x \neq y \in X$. Then there exists a coL-D_0-set U containing x with $U = U_1 - U_2$, where $U_1, U_2 \in \tau^L$; i.e., $x \in U_1$ and $x \notin U_2$. For y, we have the following cases:
(1) If $y \notin U_1$, then we are done.
(2) If $y \in U_1$ and $y \in U_2$, then U_2 contains y but not x. □

Definition 2.7. *Let (X, τ) be a space and $A \subseteq X$. The coL-closure of A in X, denoted by $\overline{A}^{coL} = \cap\{B : B \text{ is a coL-closed in } X, A \subseteq B\}$.*

Theorem 2.8. *A space (X, τ) is a coL-T_0-space if and only if for all $x \neq y \in X$, we have $\overline{\{x\}}^{coL} \neq \overline{\{y\}}^{coL}$.*

Proof. ($\Rightarrow$) Let $x \neq y \in X$. Then there exists a coL- open set U containing one point but not the other; say, $x \in U$ and $y \notin U$. Then $X - U$ is a coL-closed set containing y and $\overline{\{y\}}^{coL} \subseteq X - U$. So $x \notin \overline{\{y\}}^{coL}$. Hence $\overline{\{x\}}^{coL} \neq \overline{\{y\}}^{coL}$.
($\Leftarrow$) Let $x \neq y \in X$. Clearly, $X - \overline{\{y\}}^{coL}$ is a coc-open set containing x but not y. Hence X is coL-T_0-space. □

Definition 2.9. *A space (X, τ) is called a co-Lindelöf-T_1-space (in short, coL-T_1-space) if for all $x \neq y \in X$, there exist coL-open sets U_x, V_y with $\{U_x, V_y\} \cap \tau \neq \phi$ such that $x \in U_x$, $y \in V_y$ and $y \notin U_x$, $x \notin V_y$.*

Definition 2.10. *A space (X, τ) is called a co-Lindelöf-T_2-space (in short, coL-T_2-space) if for all $x \neq y \in X$, there exist coL-open sets U_x, V_y with $\{U_x, V_y\} \cap \tau \neq \phi$ such that $x \in U_x$, $y \in V_y$ and $U_x \cap V_y = \phi$.*

It is clear that if (X, τ) is a coL-T_1-space, then (X, τ^L) is T_1-space. In addition, every T_1-space is a coL-T_1-space. The converse need not be true as the following example shows:

Example 2.11. *Let $X = \mathbb{N}$ and $\tau = \{\phi, X\} \cup \{U_n : n \in \mathbb{N}\}$, where $U_n = \{n, n+1, n+2, \cdots\}$. Clearly, (X, τ) is not a T_1-space since for $1, 2 \in X$ each open set containing 1 must contains 2. But (X, τ) is a coL-T_1-space and a coL-T_2-space since $\tau^L = \tau$.*

Theorem 2.12. *A space (X, τ) is a coL-T_1-space if and only if every singleton is coL-closed.*

Definition 2.13. *A space (X, τ) is called co-Lindelöf-D_1-space (in short, coL-D_1-space) if for all $x \neq y \in X$, there exist coL-D-sets U_x, V_y such that $x \in U_x$, $y \in V_y$ and $y \notin U_x$, $x \notin V_y$.*

Theorem 2.14. *A coL-closed subspace of a coc-D_1-space (X, τ) is coL-D_1-space.*

Definition 2.15. *A space (X, τ) is called a co-Lindelöf-D_2-space (in short, coL-D_2-space) if for all $x \neq y \in X$, there exist disjoint coL-D- sets U_x, V_y such that $x \in U_x$, $y \in V_y$ and $y \notin U_x$, $x \notin V_y$.*

Theorem 2.16. *Let (X, τ) be a topological space. Then*

1. *If (X, τ) is a coL-T_i-space, then (X, τ) is a coL-T_{i-1}-space for $i = 1, 2$.*
2. *If (X, τ) is a coL-T_i-space, then (X, τ) is a coL-D_i-space for $i = 1, 2$.*

3. *If* (X, τ) *is a coL-*D_i*-space, then* (X, τ) *is a coL-*D_{i-1}*-space for* $i = 1, 2$.

4. *If* (X, τ) *is a coL-*D_1*-space, then* (X, τ) *is a coL-*T_0*-space.*

5. (X, τ) *is a coL-*D_1*-space if and only if* (X, τ) *is a coL-*D_2*-space .*

Proof. We will prove 5.
($\Leftarrow$) Obvious.
($\Rightarrow$) For $x \neq y \in X$, there exist coL-D-sets U_1, U_2 with $x \in U_1, y \notin U_1$ and $y \in U_2, x \notin U_1$. Assume $U_1 = V_1 - W_1, U_2 = V_2 - W_2$, where $V_1, W_1, V_2, W_2 \in \tau^L$. Then, for $x \notin U_2$, we have the following cases:
(1) $x \notin V_2$,
(2) $x \in V_2$ and $x \in W_2$.
For (1), if $x \notin V_2$, we have:
(i) $y \notin V_1$, $x \in V_1 - W_1$. Then $x \in V_1 - (V_2 \cup W_1)$ and $y \in V_2 - W_2$, so $y \in V_2 - (V_1 \cup W_2)$ and $\big(V_1 - (V_2 \cup W_1)\big) \cap \big(V_2 - (V_1 \cup W_2)\big) = \phi$.
(ii) $y \in V_1$ and $y \in W_1$, we have $x \in U_1 - U_2$, $y \in U_2$ and $\big(U_1 - U_2\big) \cap U_2 = \phi$.
For (2) If $y \in U_2 = V_2 - W_2$, $x \in W_2$ and $\big(V_2 - W_2\big) \cap W_2 = \phi$, from (1) and (2) (X, τ) is a coL-D_2- space . □

Theorem 2.17. *A space* (X, τ) *is a coL-*D_1*-space if and only if* (X, τ) *is a coL-*T_0*-space and* $int_{coL}(A_x) \neq X$ *for all* $x \in A_x \subseteq X$.

Proof. ($\Rightarrow$) For $x \in X$, there exists a coL-D-set $O_x = U - V$ with $U, V \in \tau^L$ and $x \in O_x$. But $U \neq X$. So $int_{coL}(U) \neq X$. Hence the result follows.
($\Leftarrow$) For $x \neq y \in X$, without loss of generality, there exists a coL-open set U containing x but not y and there exists coc-open set V containing y and $int_{coL}(V) \neq X$. Hence $y \in V - U$. Therefore, (X, τ) is a coLD_1-space. □

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