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Solution of Helmholtz's nonlinear differential equation and its application in Cosmology

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Abstract

In the present work, we solve the nonlinear Helmholtz differential equation by means of the Jacobi elliptic functions in the context of the problem of the deflection of light in the gravitational field of the Sun and taking into account the cosmological constant in the Einstein equations. The proposed methodology can be of great help for students and researchers who are starting the study of nonlinear problems and their applications.

1 Introduction

Nonlinear oscillators are a model that arises in various branches of physics and engineering and has been used to study various problems such as: oscillations of a large-amplitude physical pendulum, nonlinear electrical circuits, image processing, open states of DNA, the movement of satellites, Bose-Einstein condensates to mention a few [1, 2, 3, 4, 5, 6, 12].

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This work is motivated by the applications of the nonlinear oscillators and the search for analytical solutions.

We solve the nonlinear Heltmholtz differential equation analytically and exactly in the context of cosmology. We determine The angle of deflection of light in the gravitational field of the Sun taking into account the constant Cosmology.

Write the Helmholtz nonlinear differential equation as follows:

$$(\frac{du}{d\varphi})^2 = \alpha + \beta u^2 + \gamma u^3, \tag{1}$$

where α, β, γ are constants. A possible realization in the nature of equation (1) is the movement of light in the gravitational field of the Sun. This work is organized as follows:

In the first part, we obtain equation (1) in the cosmological context. In the second part, we solve equation (1) exactly by means of the Jacobi elliptic functions and the light deflection angle is determined in the gravitational field of the Sun and taking into account the cosmological constant.

2 Geodesic equations for light deflection

Equation (1) results from the geodesic path of light in the gravitational field of the Sun and from the solution of Einstein's equation [7, 8, 9] in vacuum taking into account the cosmological constant.

$$R_{\alpha\beta} - \Lambda g_{\alpha\beta} = 0, \tag{2}$$

where $R_{\alpha\beta}$ is the Ricci tensor, $g_{\alpha\beta}$ the metric tensor, Λ-cosmological constant, which will encompass the physical structure of the static universe.

Let us consider the movement of a photon in the gravitational field of a much heavier body (the Sun). The gravitational field of the center in spherical coordinates is given by the linear differential element [10]:

$$ds^2 = (1 - \frac{1}{3}r^2\Lambda - \frac{s}{r})c^2dt^2 - (1 - \frac{1}{3}r^2\Lambda - \frac{s}{r})^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3}$$

where $s = \frac{2GM}{c^2}$, G the gravitational constant, M the mass of the Sun and c is the speed of light. With the metric (3) we proceed to calculate the Christoffel symbols $\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})$ and using the equations of geodesic lines

$\frac{d^2x^\sigma}{d\tau^2} + \Gamma^\sigma_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$ and taking into account the central symmetry of our problem, any plane through the center can however be chosen as the plane $\theta = \frac{\pi}{2}$; that is, the orbit can be located in any plane through the center. We obtain the algebraic equations from the geodesics:

$$\frac{2r't'\left(3s-2\Lambda r^3\right)}{-2\Lambda r^4+6r^2-6rs}+t''=0.$$
$$\frac{c^2(t')^2\left(\Lambda r^3-3r+3s\right)\left(2\Lambda r^3-3s\right)}{18r^3}+\phi'^2\left(\frac{\Lambda r^3}{3}-r+s\right)+r''(k)-\frac{(r')^2\left(3s-2\Lambda r^3\right)}{-2\Lambda r^4+6r^2-6rs}=0. \quad (4)$$
$$\frac{2r'\phi'}{r}+\phi''=0.$$

and so we get the equation:

$$(\frac{du}{d\phi})^2=\frac{1}{b^2}+\frac{\Lambda}{3}-u^2+\frac{2GM}{c^2}u^3, \quad (5)$$

where $u=\frac{1}{r}, r$ represents the radial coordinate and b isthe impact parameter.

3 Solution of the equation by Jacobi elliptic functions

Let's write equation (5) in the following way:

$$(\frac{du}{d\phi})^2-\alpha u^3+\beta u^2-\gamma-\delta=0, \quad (6)$$

where $\alpha=\frac{2GM}{c^2}$, $\beta=1, \gamma=\frac{1}{b^2}, \delta=\frac{\Lambda}{3}$ are constants.

We look for the solution of equation (6) through the following identity:

$$u=A+B\mathrm{cn}\left(\sqrt{w}\phi\middle|\,m\right)^2, \quad (7)$$

where A, B, m, ϕ are constants to be determined. Substituting (7) into (6), we obtain:

$$-\alpha\left(A+B\mathrm{cn}\left(\sqrt{w}\phi\middle|\,m\right)^2\right)^3+\beta\left(A+B\mathrm{cn}\left(\sqrt{w}\phi\middle|\,m\right)^2\right)^2+4B^2w\mathrm{cn}\left(\sqrt{w}\phi\middle|\,m\right)^2\mathrm{dn}\left(\sqrt{w}\phi\middle|\,m\right)^2\mathrm{sn}\left(\sqrt{w}\phi\middle|\,m\right)^2-\gamma-\delta= \quad (8)$$

Taking into account:

$$\begin{aligned} sn^2(\sqrt{m}\phi,m)+cn^2(\sqrt{m}\phi,m) &= 1 \\ dn^2(\sqrt{m}\phi,m)+m^2(1-cn^2(\sqrt{m}\phi,m) &= 1, \end{aligned} \quad (9)$$

we obtain

$$\begin{aligned} &-\alpha A^3+A^2\beta-3\alpha A^2B\mathrm{cn}\left(\sqrt{w}\phi\middle|\,m\right)^2-3\alpha AB^2\mathrm{cn}\left(\sqrt{w}\phi\middle|\,m\right)^4+2A\beta B\mathrm{cn}^2-\alpha B^3\mathrm{cn}\left(\sqrt{w}\phi\middle|\,m\right)^6+ \\ &\beta B^2\mathrm{cn}^4-4B^2m^2w\mathrm{cn}\left(\sqrt{w}\phi\middle|\,m\right)^6+8B^2m^2w\mathrm{cn}\left(\sqrt{w}\phi\middle|\,m\right)^4-4B^2m^2w\mathrm{cn}\left(\sqrt{w}\phi\middle|\,m\right)^2- \\ &4B^2w\mathrm{cn}\left(\sqrt{w}\phi\middle|\,m\right)^4+4B^2w\mathrm{cn}\left(\sqrt{w}\phi\middle|\,m\right)^2-\gamma-\delta=0. \end{aligned} \quad (10)$$

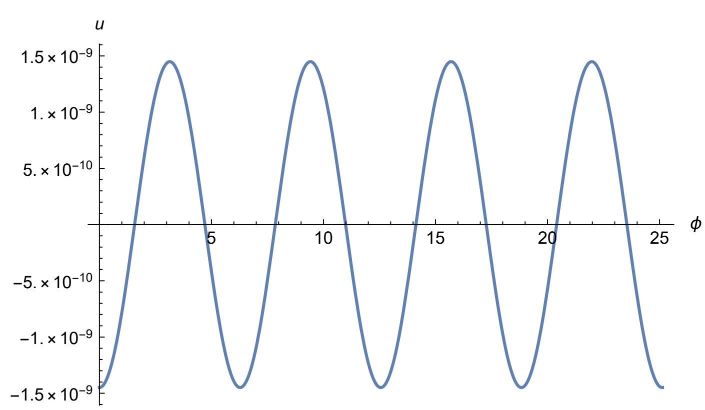
As a result, we deduce the following system of nonlinear algebraic equations:

$$\begin{aligned} &-\alpha A^3 + A^2\beta - \gamma - \delta = 0 \\ &-3\alpha A^2 B + 2A\beta B - 4B^2 m^2 w + 4B^2 w = 0 \\ &-3\alpha A B^2 + \beta B^2 + 8B^2 m^2 w - 4B^2 w = 0 \\ &-\alpha B^3 - 4B^2 m^2 w = 0 \end{aligned} \tag{11}$$

In IS units and taking into account that $\alpha = \frac{2GM}{c^2} = 2,97 \times 10^3, \beta = 1, \gamma = \frac{1}{b^2} = 2,1 \times 10^{-18}, \delta = 2,2 \times 10^{-38}$ [11], we have:
$A = 1,449 \times 10^{-9}, B = -2,898 \times 10^{-9}, w = 0,5, m = -0,0029$. Substituting in (7), we get the exact solution of the differential equation(6):

$$u(\phi) = 1.449 \times 10^{-9} - 2.898 \times 10^{-9} cn^2(0.5\phi, -0.0029) \tag{12}$$

The graph corresponding to the equation (12) is:



Graph of $u = u(\phi)$

Away from the center of the massive object, the solution tends asymptotically in Minkowski space, so that for $r -> \infty, u -> 0$, the trajectory tends to a straight line. Consequently, the angle δ of the deflection of light is the angle between these two lines, which is given by the sum of the angles ϕ_1 and ϕ_2. From the equation, $u(\phi) = 0$. From(12) and the graph, we obtain $u = 0$, if, $\phi = 1,57079$ and $\phi = 4,71239$ and so $\phi_1 = \frac{\pi}{2} - 1,57079 = 6,32679 \times 10^{-6}$ and $\phi_2 = 4,71239 - \frac{3\pi}{2} = 1,01962 \times 10^{-6}$. Therefore,

$$\delta = \phi_1 + \phi_2 = 7,34641 \times 10^{-6} = 1,5305 arcseg \tag{13}$$

4 Conclusions

We solved the differential equation that describes the behavior of light in the gravitational field of the Sun exactly. As a result, we observed that the light in the gravitational field of the Sun presents a deviation, demonstrating once more that the gravitational field around a central mass presents a curvature, establishing the prediction made by Einstein in the general theory of relativity. It should be noted that the cosmological constant Λ plays an important role in establishing the angle of deflection of light δ and, as a consequence, the curvature of space-time around a gravitational mass.

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References

[1] A.H. Nayfeh, D.T. Mook., Nonlinear oscillations, John Wiley, New York, 1995.

[2] A.I. Maimistov., Propagation of an ultimately short electromagnetic pulse in a nonlinear medium described by the fifth-order Duffing model, Opt. Spectros, **94,** no. 2, (2003), 251–257.

[3] S. Morfa, J.C. Comte, A nonlinear oscillators network devoted to image processing, International Journal of Bifurcation and Chaos, **14,** no. 4, (2009), 1385–1394.

[4] A.H. Salas, J.E. Castillo, Exact solutions for a nonlinear model, Applied Mathematics and Computation, **217,** (2010), 16461651

[5] A.H. Salas, J.E. Castillo, La ecuación Seno-Gordon perturbada en la Dinámica no lineal del ADN, Revista Mexicana de Física, **58,** (2012), 481–487.

[6] A.H. Salas, Castillo H., Jairo E. Exact solutions to cubic duffing equation for a nonlinear electrical circuit, Visión Electrónica: Algo Más Que Un Estado Sólido, **8,** fasc. 1, (2014), 46–53.

[7] A. Einstein, Kosmologische Betrachtugen, Zur allgemeinen Relativitatstheorie, Sitzungsher. Preuss. Akad. Wiss., (1917), 142–152.

[8] A. Einstein, Kosmologische Problem der allgemeinen Relativitatstheorie, Sitzungsher. Preuss. Akad. Wiss., phys-math., k1., (1931), 235–237.

[9] A. Einstein, On the cosmological problem. The meaning of relativity, 2nd edition, Princeton, 1945.

[10] Friedrich Kottler, Annalen Der Phydik., **14,** (1918), 401–462.

[11] A.H. Salas, J.E. Castillo Hernandez, J.E. Pinzon Quintero, Calculation of the Cosmological constant for the Planetary System in Schwarzschild's Cosmological Model, Universe, **8,** (2022), 449. https://doi.org/10.3390/universe8090449

[12] Jairo E. Castillo, Jamel Castillo Mejia, Alvaro H. Salas, A Nonlinear Transmission Model with Analytical Soliton Solution, International Journal of Mathematics and Computer Science, **18,** no. 1, (2023), 119–129.